TARGET-BASED OPTIMIZATION IN OPERATIONS MANAGEMENT

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DECLARATION

I hereby declare that the thesis is my original

work and it has been written by me in its entirety.

I have duly acknowledged all the sources of
information which have been used in the thesis.

This thesis has also not been submitted for any
degree in any university previously.

Long, Zhuoyu
21 May 2013

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ABSTRACT

In this thesis, we investigate the decision criteria for two classical problems in operations management, inventory control and project management, by taking into account the effect of aspiration level such as profit target. Different to the existing approach that maximizes the probability of the profit reaching targets, we optimize a new target-oriented decision criterion. In inventory management, we study both single-period and multiple-period problems. For the single-period (newsvendor) problem, the results from our theoretical model happen to be consistent with existing findings in newsvendor experiments. For the multi-period problem, we incorporate the financing decisions, lending/borrowing activities, to smooth out consumptions over time. We show that if borrowing and lending are unrestricted, the optimal financing policy derived from the target-based criterion is to finance consumptions at the target levels for all periods except the last. Moreover, the optimal inventory policy preserves the structure of base-stock policy or (s,S) policy, and could be achieved with relatively modest computational effort. Under restricted financing, we show that the optimal policies are indeed as the same as those that maximize expected additive-exponential utilities, and can be obtained by an efficient algorithm. In project management, we consider a project selection problem where each project has uncertain return with partially characterized probability distribution. The model captures correlation and interaction effects such as synergies. We solve the model using binary search, and obtain solutions of the subproblems from Benders decomposition techniques. As a simple alternative, we describe a greedy heuristic, which routinely provides project portfolios with near optimal underperformance risk.

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1. INTRODUCTION

In operations management models involving uncertainties, we typically assumes that the decision maker is risk neutral and maximizes the expected profit, or equivalently, minimizes the expected cost. Although simple and elegant, this assumption neglects the risk embedded in these problems, where the risk is not always ignorable, especially when the scenarios would not be repeated for a large number of times. Take start-up companies for example, a decision subject to tremendous loss in the case of unfavorable uncertainty realization can be devastating and lead to bankruptcy.

To take into account the decision maker's risk attitude, researchers have explored alternative normative models such as maximizing expected utility or minimizing a risk measure. With different shapes of utility functions, or different forms of risk measure, the risk attitudes of decision makers are captured in a more general way compared with the risk neutral model. These researches, however, still ignore one important factor in decision making process, which is the target. In this thesis, we investigate how to make optimal decisions in the presence of a target profit in classical operations management problems. In particular, we start from the newsvendor problem, which serves as a building block for inventory theory. After that, we analyze the general dynamic programming problem, and apply our framework on the multiple period inventory-pricing problem. Besides, we also study the target-based framework under the zero-one optimization setting, investigate the project selection problem.

Structure of the chapter. In Section 1.1, we discuss the motivations for incorporating targets in operations management problems and provide related literatures. Section 1.2 presents the outline of the thesis.

1.1 Motivation and Literatures Review

It is a common phenomenon in industry that, in making decisions, managers are often concerned about a profit target to reach. The concern of profit target is driven by several reasons.

First, as Conger et al. (1998) and Bossidy (2007) point out, one of the key aspects in performance appraisal, which affects managers' promotion, bonus, and many other interests, is related to the achievement of certain targets. Further, Hirsch (1994, p.609) suggests that firms need to assess "how managers are achieving the goals and objectives of the company rather than how they might be optimizing some local measures."

Second, at the firm level, the external evaluation of the firm's performance is also largely dependent upon the achievement of targets. Take a company's stock price for example, it is widely believed that it depends on the company's ability to meet its financial goals (Rappaport 1999). A classical case involves Ebay in the fourth quarter of 2004. Ebay reported earnings of 23 cents per share, which missed the target of 24 cents per share. After the report was issued, Ebay's stock tumbled more than 11% within a few hours (CNNMoney 2005). What happens to Swiss Life Holding, the largest life insurer in Switzerland, is another example. In 2008, it failed to achieve the profit target of $1.6 billion. As a result, its share price was decreased by 20% in Zurich trading (Giles 2008).

Last, setting targets is considered to be helpful in improving employees' performance. Compared to "do their best," people in general would work more affirmatively and exert more endeavor to attain a reasonable target (Locke and Latham 2002, Rasch and Tosi 1992, Barrick et al. 1993). In fact, it is such a strong relationship that Locke and Latham (2002) consider goal setting as possibly the best managerial tool in terms of effectiveness.

The prevalence of setting and meeting targets in decision making is well observed. Through interviews with twenty companies, Lanzillotti (1958) show that most of these firms set their goals to achieve a target profit. In another interview conducted by Mao (1970), he establishes that managers view risk as the probability of meeting a target profit. Brown and Tang (2006) also demonstrate that when placing orders, inventory managers are concerned about the ability to attain a target profit. Laboratory experiments, albeit not in the newsvendor problem setting, have also been run to illustrate the impact of target on decision making (Payne et al. 1980, 1981). Further, the importance of incorporating a target into decision making is highlighted in Simon (1955), Rubinstein (1998), and Gigerenzer and Selten (2002).

Motivated by the evidence above, we aim to investigate the decision making in operations management problem under the consideration of a target profit.

1.2 Structure of the Thesis

The rest of the thesis is organized as follows.

- **Chapter 2: The Impact of a Target on Newsvendor Decisions.** We investigate the impact of a target on newsvendor decisions. Different from the existing approach that maximizes the probability of the profit reaching the target, in this chapter we model the effect of a target by maximizing the satisficing measure of a newsvendor's profit with respect to that target. We study two satisficing measures: i) CVaR satisficing measure that evaluates the highest confidence level of CVaR achieving the target; and ii) Entropic satisficing measure that assesses the smallest risk tolerance level under which the certainty equivalent for exponential utility function achieves the target. For both satisficing measures, we find that the optimal ordering quantity increases with the target level. Further, the newsvendor orders more than the risk-neutral solution (over-order) sometimes and less than that (under-order) other times, depending on the target level. The more interesting finding is that if the target is proportional to the unit marginal profit and is also determined by only one other demand-related factor, then the newsvendor over-orders low-profit product and under-orders high-profit product.

- **Chapter 3: Managing Operational and Financing Decisions to Meet Consumption Targets.** We study dynamic operational decision problems where risky cash flows are being resolved over a finite planning horizon. Financing decisions via lending and borrowing are available to smooth out consumptions over time with the goal of achieving some prescribed consumption targets. Our target-oriented decision criterion is based on the aggregation of Aumann and Serrano (2008) riskiness indices of the consumption excesses over targets, which has salient properties of subadditivity, convexity and respecting second-order stochastic dominance. We show that if borrowing and lending are unrestricted, the optimal policy based on this criterion is to finance consumptions at the target levels for all periods except the last. Moreover, the optimal policy has the same control structure as the optimal risk neutral policy and could be achieved with relatively modest computational effort. Under restricted financing, we show that for convex dynamic decision problems, the optimal policies are indeed as the same as those that maximize expected additive-exponential utilities, and can be obtained by an efficient algorithm. We also analyze the optimal policies of joint inventory-pricing decision problems under the target-oriented criterion and provide optimal policy structures. With a numerical study for inventory control problems, we report favorable computational results for using targets in regulating uncertain consumptions over time.

- **Chapter 4: Managing Underperformance Risk in Project Portfolio Selection.** We consider a project selection problem where each project has an uncertain return with partially characterized probability distribution. The decision maker selects a feasible subset of projects so that the risk of the portfolio return not meeting a specified target is minimized. Our work extends the riskiness index of Aumann and Serrano (2008) by incorporating the target and also distributional ambiguity. We minimize the underperformance risk of the project portfolio, which we define as the reciprocal of the absolute risk aversion (ARA) of an ambiguity averse individual with constant ARA who is indifferent between the target return with certainty and the uncertain portfolio return. Our model captures correlation and interaction effects such as synergies. We solve the model using binary search, and obtain solutions of the subproblems from Benders decomposition techniques. A computational study shows that project portfolios generated by minimizing the underperformance risk have certain advantages in achieving the target compared with those found by benchmark approaches, including maximization of expected return, minimization of underperformance probability, mean-variance analysis, and maximization of Roy's (1952) safety first ratio. As a simpler alternative, we describe a greedy heuristic, which routinely provides project portfolios with near optimal underperformance risk.

- **Chapter 5: Conclusions.** This chapter provides the conclusion of the thesis, which summarizes key findings and highlights future research.

1.3 Notation

Throughout this thesis, we denote a random variable by a character with the tilde sign such as $\tilde{z}$, and z is its realization. A random variable is defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ where Ω is the set of possible outcomes, $\mathcal{F}$ is a

σ-algebra that describes the set of all possible events, and $\mathbb{P}$ is the probability measure function.

2. THE IMPACT OF A TARGET ON NEWSVENDOR DECISIONS

To capture the impact of a target on newsvendor decision making as well as address the drawback of attainment probability measure or expected utility, our model adopts the recently developed satisficing measure (Brown and Sim 2009, Brown et al. 2012), a class of risk measures that evaluate the ability of a certain metric – which is associated with the underlying random payoff – achieving a target. The attainment probability measure used in Lau (1980) is in fact a special case of a satisficing measure with the metric being the quantile. However, our study focuses on satisficing measures with other metrics such as those taking into account the magnitude of unfavorable profit realization. We focus on two commonly used metrics with respect to the random profit: CVaR (Rockafellar and Uryasev 2000, 2002) and Certainty Equivalent for exponential utility function (Mas-Collel et al. 1995). CVaR measures the expected value of the profit that is falling below a certain quantile value (we call it worst-case-scenario expected profit, where "worst" is associated with confidence level). To incorporate the fact that people are not always risk averse (Kahneman and Tversky 1979), we extend the definition of CVaR such that it also measures the expected value of the profit that is above a certain quantile value (we call it best-case-scenario expected profit, where "best" is also associated with confidence level). The Certainty Equivalent for a risky alternative is the certain amount that is equally preferred to the alternative. Similarly, we study the certainty equivalent for both the risk-averse and risk-seeking scenarios. Corresponding to these two metrics, we consider CVaR Satisficing Measure (CSM) and Entropic Satisficing Measure (ESM), respectively. The former evaluates the confidence level of CVaR achieving the target and the latter assesses the risk tolerance level under which the certainty equivalent achieves the target. Note that it is desirable to have a CSM value as big as possible. This suggests that one can be highly confident about the expected profit achieving the target even if the random profit is realized in an undesirable region. Similarly, higher ESM value is preferred because it implies that a highly conservative decision maker can still have the certainty equivalent exceeding the target and accept the underlying decision. As such, the objective of the newsvendor is to find an order quantity that maximizes the CVaR (and Entropic) satisficing measure.

It is worth noting that CSM and ESM represent two different ways decision makers perceive risks. Since CVaR measures the expectation conditioning on falling below (or above, in our extended definition) a certain quantile, CSM reflects the emphasis on downside (or upside) risk. The ESM, however, is based on the certainty equivalent for exponential utility function and takes into account all realizations of underlying randomness. As such, ESM captures the attention on full scale risk. Interestingly, regardless which of the two measures is adopted, our findings on the effect of target remain the same, which suggests that our target-based newsvendor model is robust to how decision makers recognize risks.

Before presenting the models and analyses, we summarize our contributions to literature.

- We build an easy-to-apply normative model to capture the effect of target on newsvendor decision. With the decision criteria of both CSM and ESM, we are able to provide a more comprehensive analysis than existing literature on the impact of target. Our results complement that of the existing literature that only maximizes the probability of profit reaching the target.

- We characterize the optimal ordering strategy for target-based newsvendors and show that (i) optimal order quantity increases with target level; (ii) the same decision maker can sometimes order more than and other times less than the risk neutral ordering quantity (i.e., the one maximizing expected profit), depending on the target level; and (iii) if the target is proportional to the unit marginal profit, the newsvendor will under-order high-profit products, and over-order low-profit products.

- We take one step further and show that if the target is set properly, our model gives exactly the same solution as the expected utility model does.

Structure of the chapter. Section 2.1 describes the CVaR satisficing measure, illustrates how to find the optimal order quantity under CVaR satisficing measure, and also shows how the optimal order quantity is affected by the target. Section 2.2 investigates the newsvendor problem under Entropic satisficing measure. Section 2.3 presents computational studies to compare the performance of our target-based newsvendor decisions to that of other newsvendor model decisions. We conclude this chapter in Section 2.4. Finally, in Section 2.5, we provide several lemmas which are needed for the proof of some theorems in this chapter.

2.1 Newsvendor Decision with CVaR Satisficing Measure

A newsvendor decides how many units of product to order before the selling season. Each unit can be purchased at cost c and sold at price p. The random demand $\tilde{d}$ is assumed to be bounded by $[\underline{d}, \bar{d}] \subseteq \Re_+$ and without loss of generality, continuously distributed. Note that all the results in this chapter can be easily extended to general demand distribution, which can be unbounded and not necessary continuous.

For simplicity, the unsatisfied demand is lost and the salvage value for unsold items is assumed to be zero. With an order quantity y, the newsvendor's profit is given by:

$$\tilde{v}(y) = -cy + p\min(y, \tilde{d}). \tag{2.1}$$

2.1.1 CVaR Satisficing Measure

Let $\mathcal{B}$ be the set of bounded random variables. Following the path of the recently developed CVaR satisficing measure (Brown et al. 2012), which is to quantify a random profit's risk with respect to a specified target, we define CVaR Satisficing Measure as follows:

Definition 1. Given a target profit $\tau \in \Re$, the CVaR satisficing measure (CSM), $\rho_\tau : \mathcal{B} \to [-1, 1]$ is defined as:

$$\rho_\tau(\tilde{v}) = \begin{cases} \sup\left\{\eta \in (-1,1) : CVaR_\eta(\tilde{v}) \geq \tau\right\}, & \text{if } feasible, \\ -1 & \text{otherwise,} \end{cases} \tag{2.2}$$

where $CVaR_\eta : \mathcal{B} \to \Re$ is defined as:

$$CVaR_\eta(\tilde{v}) = \begin{cases} \max\limits_{a \in \Re} \left\{a + \frac{1}{1-\eta}\mathbb{E}[\min\{\tilde{v} - a, 0\}]\right\} & \text{if } \eta \in [0,1), \\ -CVaR_{-\eta}(-\tilde{v}) & \text{if } \eta \in (-1,0). \end{cases} \tag{2.3}$$

It is worth mentioning that if $\tilde{v}$ is continuously distributed, an equivalent but much more intuitive definition for $CVaR_\eta$ is:

$$CVaR_\eta(\tilde{v}) = \begin{cases} \mathbb{E}\left[\tilde{v} \middle| \tilde{v} \leq q_{1-\eta}(\tilde{v})\right], & \text{if } \eta \in [0,1), \\ \mathbb{E}\left[\tilde{v} \middle| \tilde{v} \geq q_{-\eta}(\tilde{v})\right], & \text{if } \eta \in (-1,0), \end{cases} \tag{2.4}$$

where $q_\eta(\tilde{v})$ is the unique η-quantile of $\tilde{v}$. According to the definition in (2.4), for $\eta \in [0, 1)$, $CVaR_\eta(\tilde{v})$ measures the expectation of $\tilde{v}$ in the worst $(1 - \eta)$ case realizations; for $\eta \in (-1, 0)$, it measures the expectation in the best $(1 + \eta)$ case realizations.

The traditional $CVaR_\eta$ is defined only on $\eta \in [0, 1)$ and measures the worst case expectation, which implies risk averse preference. To capture the risk seeking behavior, we enable $CVaR_\eta$ to assess the best case performance by extending the range of η to include $(-1, 0)$. As such, with $\eta < 0$, $CVaR_\eta$ results in risk seeking choice due to its nature of seeking for best case performance. While $CVaR_\eta$ is a convex risk measure and favors diversification for $\eta > 0$ (Rockafellar and Uryasev 2000, 2002), we can verify that it is a concave risk measure and favors concentration for $\eta < 0$.

By Definition 1, CSM measures the highest η that guarantees $CVaR_\eta$ achieving a target. Observe that $CVaR_\eta$ is essentially a conditional expectation of the random payoff. The index η prescribes the condition for this conditional expectation: a positive η implies that the expectation is taken over the worst $(1 - \eta)$ case, whereas a negative η suggests an expectation w.r.t. to the realization of the best $(1 + \eta)$ case. As such, it is desirable for a random payoff to have a high CSM value, as this implies that the random payoff $\tilde{v}$ is more secure w.r.t. τ. Intuitively, we can think of ρ_τ as a security index for random payoff to achieve target.

To further illustrate the concept of CSM, assume that for a continuous random variable $\tilde{v}$, we have $\rho_\tau(\tilde{v}) = k$, where $k \in (0, 1)$. By definition of ρ_τ we know that $\mathbb{E}[\tilde{v}|\tilde{v} \leq q_\lambda(\tilde{v})] < \tau$ if and only if $\lambda < 1 - k$. In other words, conditioning on that the randomness does not always realize in the worst $(1 - k)$ case, the expectation of $\tilde{v}$ will exceed the target. Similarly, if $\rho_\tau(\tilde{v}) = -k < 0$, then $\mathbb{E}[\tilde{v}|\tilde{v} \geq q_\lambda] \geq \tau$ if and only if $\lambda \geq k$. That is, $\mathbb{E}[\tilde{v}]$ will be no less than τ conditioning on that realization of the randomness will surely fall in the best $(1 - k)$ case.

2.1.2 Newsvendor with CSM

With the framework of CSM, the newsvendor problem with target profit τ is:

$$\rho_\tau^* = \max_{y \geq 0} \rho_\tau\left(\tilde{v}(y)\right) \tag{2.5}$$

where $\tilde{v}(y)$ is given by (2.1). For this problem, an order quantity needs to be decided to maximize CSM, which means that this optimal order quantity should be the one that makes it most secure for the profit to achieve the target.

If problem (2.5) has an optimal objective value in $(-1, 1)$, it can be reformulated as:

$$\begin{aligned} \max \quad & \eta \\ \text{s.t.} \quad & CVaR_\eta\left(\tilde{v}(y)\right) \geq \tau, \\ & \eta \in (-1, 1), \\ & y \geq 0. \end{aligned} \tag{2.6}$$

By the definition in (2.3), we can verify that $CVaR_\eta(\cdot)$ is non-increasing in η. Therefore, we can find the optimal solution for the problem in (2.6) by performing a binary search on η. For each $\eta \in (-1, 1)$, we need to solve the following subproblem:

$$\max_{y \geq 0} CVaR_\eta(\tilde{v}(y)). \tag{2.7}$$

Note that the optimal value of the problem in (2.5) is 1 if and only if for all $\eta \in (-1, 1)$, we have the optimal value of the problem in (2.7) no less than τ; and it is -1 if and only if we have the optimal value of the problem in (2.7) strictly less than τ for all $\eta \in (-1, 1)$. As such, *we can efficiently solve* *the problem in (2.5) with a binary search on η as long as we are able to solve the problem in (2.7)* easily, which indeed is the case as suggested by Lemma 1 below that provides the solution to (2.7).

Lemma 1. For any $\eta \in (-1, 1)$, we have

$$\arg\max_{y \geq 0} CVaR_\eta\left(\tilde{v}(y)\right) = \begin{cases} F^{-1}\left(\xi - \eta\xi\right) & \text{if } \eta \in [0, 1), \\ F^{-1}\left(\xi - \eta(1-\xi)\right) & \text{if } \eta \in (-1, 0), \end{cases}$$

where $\xi = \frac{p-c}{p}$ is called critical fractile, and F is the cumulative distribution of $\tilde{d}$.

Proof. For the case of $\eta \in [0, 1)$, the result can be referred to Gotoh and Takano (2007). Here we just discuss on the case of $\eta \in (-1, 0)$, where

$$\begin{aligned} CVaR_\eta\left(\tilde{v}(y)\right) &= -CVaR_{-\eta}\left(-\tilde{v}(y)\right) \\ &= -\max_{a \in \Re}\left\{a + \frac{1}{1+\eta}\mathbb{E}\left[\min\{-\tilde{v}(y) - a, 0\}\right]\right\} \\ &= \min_{a \in \Re}\left\{a + \frac{1}{1+\eta}\mathbb{E}\left[\left(p\min(y, \tilde{d}) - cy - a\right)^+\right]\right\} \\ &= \min_{a \in \Re} g(y, a). \end{aligned}$$

Here the $g(y, a)$ is defined as

$$\begin{aligned} g(y, a) &= a + \frac{1}{1+\eta}\mathbb{E}\left[\left(p\min(y, \tilde{d}) - cy - a\right)^+\right] \\ &= a + \frac{1}{1+\eta}\left(\int_0^y (pz - cy - a)^+ \, dF(z) + (py - cy - a)^+ \left(1 - F(y)\right)\right). \end{aligned}$$

For any given $y \geq 0$, we discuss on the three different cases.

1. $a \leq -cy$. In this case, $g(y,a) = a + \frac{1}{1+\eta}\mathbb{E}\left[(\tilde{v}(y) - a)\right]$, $\frac{\partial g}{\partial a} = \frac{\eta}{1+\eta} < 0$.

2. $-cy \leq a \leq py - cy$. In this case, we have

$$\begin{aligned} g(y,a) &= a + \frac{1}{1+\eta}\left(\int_{\frac{cy+a}{p}}^{y}(pz - cy - a)\,dF(z) + (py - cy - a)\,(1 - F(y))\right), \\ \frac{\partial g}{\partial a} &= \frac{F\left(\frac{cy+a}{p}\right) + \eta}{1+\eta}. \end{aligned}$$

3. $a \geq py - cy$. In this case, $g(y,a) = a$, $\frac{\partial g}{\partial a} = 1 > 0$.

Therefore, let $a^*(y) = \arg\min_{a \in \Re} g(y,a)$, we should have $-cy \leq a^*(y) \leq py - cy$. Hence, it suffices to consider $a \in [-cy, py - cy]$ in the following discussion, and it implies $(cy + a)/p \in [0, y]$.

If $y \leq F^{-1}(-\eta)$, $\frac{\partial g}{\partial a} \leq \frac{F(y)+\eta}{1+\eta} \leq 0$, $py - cy = a^*(y)$, $CVaR_\eta(\tilde{v}(y)) = py - cy$, and

$$\frac{\partial CVaR_\eta(\tilde{v}(y))}{\partial y} = p - c > 0.$$

If $y \geq F^{-1}(-\eta)$, by FOD, $a^*(y) = pF^{-1}(-\eta) - cy$,

$$CVaR_\eta(\tilde{v}(y)) = -cy + \frac{1}{1+\eta}\left(\int_{F^{-1}(-\eta)}^{y} pz dF(z) + py(1 - F(y))\right),$$

$$\frac{\partial CVaR_\eta(\tilde{v}((y))}{\partial y} = -c + \frac{p(1 - F(y))}{1+\eta}.$$

By FOD, the maximizer of $CVaR_\eta(\tilde{v}((y))$ is $y^* = F^{-1}\left(1 - \frac{c}{p}(1+\eta)\right) = F^{-1}\left(\xi - \eta(1-\xi)\right)$. $\square$

Remark: As we assume the demand is continuously distributed, F^{-1} is a mapping to a scalar. Hence, by Lemma 1 we can see that the problem (2.7) has unique optimal solution when $\eta \in (-1, 1)$. Suppose we relax the assumption on the random demand such that it can follow non-continuous distribution, then F^{-1} may possibly map to a set instead of a scalar, in which case the solution for the problem (2.7) is no longer unique and that complicates the following analysis. While we have proved all the following results still hold for non-continuous distribution, here we assume the demand is continuously distributed to simplify the analysis.

We now proceed to examine how the target profit affects the ordering decision.

Theorem 1. Assume that $\tau_1 \geq \tau_2$. Then we have: 1) $\rho^*_{\tau_1} \leq \rho^*_{\tau_2}$; and 2) $\exists y_1 \geq y_2 \geq 0$ such that $y_i \in \arg\max_{y \geq 0} \rho_{\tau_i}(\tilde{v}(y))$, $i \in \{1, 2\}$.

Proof. For $i \in \{1, 2\}$, denote $\rho_i = \rho^*_{\tau_i} = \max_{y \geq 0} \rho_{\tau_i}(\tilde{v}(y))$. By the definition of CSM we can get $\rho_1 \leq \rho_2$ since $\tau_1 \geq \tau_2$.

Note that $\forall y \in [0, \underline{d}]$, $\mathbb{P}(\tilde{v}(y) \leq \tilde{v}(\underline{d})) = 1$; and $\forall y \in [\bar{d}, \infty)$, $\mathbb{P}(\tilde{v}(y) \leq \tilde{v}(\bar{d})) = 1$. Hence, there must exist $y \in [\underline{d}, \bar{d}]$ maximizing CSM. Here we just look at the existence of $y_i \in [\underline{d}, \bar{d}]$ to prove the result.

First, we consider the case that $-1 < \rho_1 \leq \rho_2 < 1$. Let

$$y_i = \arg\max CVaR_{\rho_i}(\tilde{v}(y)) = \begin{cases} F^{-1}(\xi - \rho_i \xi) & \text{if } \rho_i \in [0, 1), \\ F^{-1}(\xi - \rho_i(1 - \xi)) & \text{if } \rho_i \in (-1, 0). \end{cases} \tag{2.8}$$

By definition, we can easily check that $\rho_{\tau_i}(\tilde{v}(y_i)) = \rho_i$, $y_i \in \arg\max_{y \geq 0} \rho_{\tau_i}(\tilde{v}(y))$.

By (2.8), we have $y_1 \geq y_2$ since $\rho_1 \leq \rho_2$.

Secondly, consider the case that $\rho_1 = -1$. We have $\rho_{\tau_1}(\tilde{v}(y)) = -1$ for all y. Choose $y_1 = \bar{d}$. For any $y_2 \in [\underline{d}, \bar{d}]$ such that $\rho_{\tau_2}(\tilde{v}(y_2)) = \rho_2$, we have $y_1 \geq y_2$.

Finally, consider the case that $-1 < \rho_1 \leq \rho_2 = 1$. Let $y^* \in [\underline{d}, \bar{d}]$ be an order quantity such that $\rho_{\tau_2}(\tilde{v}(y^*)) = 1$. By Lemma 2, $\mathbb{P}(\tilde{v}(y^*) \geq \tau_2) = 1$, which implies $-cy^* + p\underline{d} \geq \tau_2$. Hence, we have $-c\underline{d} + p\underline{d} \geq \tau_2$, and $\rho_{\tau_2}(\tilde{v}(\underline{d})) = 1$. Choose $y_2 = \underline{d}$. For any $y_1 \in [\underline{d}, \bar{d}]$ such that $\rho_{\tau_1}(\tilde{v}(y_1)) = \rho_1$, we have $y_1 \geq y_2$. $\square$

Remark: $\max\limits_{y \geq 0} \rho_{\tau_i}(\tilde{v}(y))$, $i \in \{1, 2\}$ may not necessary has unique optimal solution. Therefore, in the above theorem, we use $y_i \in \arg\max\limits_{y \geq 0} \rho_{\tau_i}(\tilde{v}(y))$ rather than $y_i = \arg\max\limits_{y \geq 0} \rho_{\tau_i}(\tilde{v}(y))$ for $i \in \{1, 2\}$.

The first part of Theorem 1 shows that the newsvendor's maximal CSM decreases with the target profit. This is because the same random profit must be more secure if we have a lower target, and be riskier if we have a higher target. Consequently, the best quantity decision made under a low target must make the profit at least as secure as that under a high target.

The second part of Theorem 1 suggests that the newsvendor will order more if the target is higher. To understand this result, we note that a high target is an indication of the newsvendor's soaring ambition, which is more likely to be realized if the newsvendor places a larger order. Let us consider the extreme cases. Assume that the target is $\tau = (p - c)\underline{d}$. Then this target can be achieved for sure if the newsvendor orders $\underline{d}$. Hence, $\underline{d}$ is the most secure order quantity. On the other hand, if the target τ is very high such that the random profit from ordering small quantities is always strictly less than it, then the newsvendor can do nothing but place a large order.

Let $y_N = F^{-1}(\xi)$ be the risk neutral newsvendor solution, which maximizes the expected profit. We then have the following theorem.

Theorem 2. If $\tau = \mathbb{E}\left[\tilde{v}\left(y_N\right)\right]$, then $y_N \in \arg\max_{y\geq 0} \rho_\tau\left(\tilde{v}(y)\right)$.

Proof. By the definition in (2.4), $CVaR_0\left(\tilde{v}\left(y_N\right)\right) = \mathbb{E}\left[\tilde{v}\left(y_N\right)\right] = \tau$. Therefore, $\rho_\tau\left(\tilde{v}\left(y_N\right)\right) \geq 0$.

Since $\tilde{d}$ is continuously distributed, $\mathbb{E}[\tilde{v}(y)]$ is uniquely maximized at y_N. Therefore, for any $y \geq 0$ and $y \neq y_N$, we have $CVaR_0\left(\tilde{v}\left(y\right)\right) = \mathbb{E}\left[\tilde{v}\left(y\right)\right] < \mathbb{E}\left[\tilde{v}\left(y_N\right)\right] = \tau$. That implies $\rho_\tau\left(\tilde{v}(y)\right) \leq 0 \leq \rho_\tau\left(\tilde{v}\left(y_N\right)\right)$. □

Theorem 2 says that if the newsvendor's target is the maximal expected profit, then the risk-neutral newsvendor solution gives the highest CSM. This is intuitive because for any other order quantity, the risk neutral expectation of profit is less than the target, $\tau = \mathbb{E}[v(\tilde{y}_N)]$. As such, to enable its CVaR to reach the target, it's only possible by looking at the best-case profit realization when $\eta < 0$, whereas the risk-neutral solution can do so for $\eta = 0$.

Corollary 1.

1. If $\tau \leq \mathbb{E}\left[\tilde{v}\left(y_N\right)\right]$, then $\exists y^* \leq y_N$ such that $y^* \in \arg\max_{y\geq 0} \rho_\tau\left(\tilde{v}(y)\right)$.
2. If $\tau \geq \mathbb{E}\left[\tilde{v}\left(y_N\right)\right]$, then $\exists y^* \geq y_N$ such that $y^* \in \arg\max_{y\geq 0} \rho_\tau\left(\tilde{v}(y)\right)$.

Proof. It follows immediately from Theorems 1 and 2. □

In the newsvendor problem, an important benchmark is the risk neutral solution, y_N. A newsvendor is said to *under-order* if she orders less than

y_N, and *over-order* if orders more than y_N. Corollary 1 shows that the newsvendor under-orders when the target is lower than the maximal expected profit, and over-orders when the target is higher than that. Fundamentally, in our model the target can influence the decision maker's risk attitude. For example, if the target is very high, then the decision maker may just take the chance and "pray for odds." However, if the target is low, then it makes more sense to be more conservative.

So far we have taken the target profit as exogenously given, without considering how it is set and what form it takes. In fact, in comparison to the substantial body of empirical research on the effect of target (e.g. Brown and Tang 2006), the research on how people form their targets is rather limited. In the best of our knowledge, the only descriptive research is a field study by Merchant and Manzoni (1989), who show that in practice the targets are usually set in a way such that they can be achieved in eighty to ninety percent of the time. The other stream of research, which can be considered as a guide on how to set targets normatively, mainly focus on how the challenging level of the goal impacts employee performance (e.g. Tubbs 1986, Locke and Latham 2002, Fried and Slowik 2004). In what follows, we first follow the path of Kőszegi and Rabin (2006) and make the assumption that the newsvendors' target profit is determined by their expectations under simple heuristics, and study the property of optimal order quantities.

Theorem 3. Assume $\tau = (p-c)\times\alpha(\tilde{d})$, where $\alpha : \mathcal{B} \to \Re_+$ is a function of the random demand. Then there exists a threshold value ζ such that if $\frac{p-c}{p} \geq \zeta$, we can find $y^* \leq y_N$ such that $y^* \in \arg\max_{y\geq 0} \rho_\tau\left(\tilde{v}(y)\right)$; and if $\frac{p-c}{p} \leq \zeta$, we can find $y^* \geq y_N$ such that $y^* \in \arg\max_{y \geq 0} \rho_\tau \left(\tilde{v}(y)\right)$.

Proof. Let $r(\xi) = \mathbb{E}[\tilde{v}(y_N)] - \tau = \mathbb{E}[\tilde{v}(F^{-1}(\xi))] - (p-c)\alpha(\tilde{d})$. According to Corollary 1, $r(\xi) \leq 0$ implies over-ordering, and $r(\xi) \geq 0$ implies under-ordering. If $\alpha(\tilde{d}) \leq \underline{d}$, we get for all ξ,

$$r(\xi) = \mathbb{E}[\tilde{v}(y_N)] - (p-c) \times \alpha(\tilde{d}) \geq \mathbb{E}[\tilde{v}(\underline{d})] - (p-c)\underline{d} = 0.$$

So we just need to choose $\zeta = 0$.

Similarly, if $\alpha(\tilde{d}) \geq \bar{d}$, we get $r(\xi) \leq 0$ for all ξ. So we just choose $\zeta = 1$.

Now we just consider $\alpha(\tilde{d}) \in \left(\underline{d}, \bar{d}\right)$. Recall that $y_N = F^{-1}(\xi)$, so we have

$$\begin{aligned} r(\xi) &= p\left(\int_{\underline{d}}^{F^{-1}(\xi)} x \cdot dF(x) + \int_{F^{-1}(\xi)}^{\bar{d}} F^{-1}(\xi) dF(x)\right) - cF^{-1}(\xi) - (p-c)\alpha(\tilde{d}) \\ &= p\left(\int_{\underline{d}}^{F^{-1}(\xi)} x \cdot dF(x) - \xi\alpha(\tilde{d})\right), \qquad (2.9) \\ r'(\xi) &= p\left(F^{-1}(\xi) - \alpha(\tilde{d})\right). \end{aligned}$$

Therefore $r(0) = 0$, $r'(0) < 0$, and $r(\xi)$ is convex since $r'(\xi)$ is increasing.

If $\alpha(\tilde{d}) < \mathbb{E}[\tilde{d}]$, we have $r(1) > 0$, and $\exists \zeta \in (0,1)$ such that $r(\xi) \leq 0$ for $\xi \leq \zeta$, and $r(\xi) \geq 0$ for $\xi \geq \zeta$.

If $\alpha(\tilde{d}) \geq \mathbb{E}[\tilde{d}]$, $r(1) \leq 0$, and $r(\xi) \leq 0$ for all possible $\xi \in [0,1]$, so we can choose $\zeta = 1$. □

According to Theorem 3, if the newsvendor develops her targets following the simple heuristic of $\tau = (p - c)\alpha(\tilde{d})$, i.e., the target is proportional to the unit marginal profit as well as a demand-related factor, she will then under-order high-profit products and over-order low-profit products. Here $\tau = (p - c) \times \alpha(\tilde{d})$ can be considered as a simple and intuitive heuristic for the newsvendors to set their targets. For example, a newsvendor can simply treat the random demand as a deterministic one with the value equal to its expectation. After taking 20% off as the cost of uncertainty, her target profit is set to be $\tau = 80\% \times (p - c)\mathbb{E}[\tilde{d}]$. Hence, for this newsvendor we have $\alpha(\tilde{d}) = 0.8\mathbb{E}[\tilde{d}]$. Likewise, the target can be $\tau = 0.6(p - c) \times m(\tilde{d})$, where $m(\tilde{d})$ is the mode of the demand distribution.

It is worth noting that high-profit and low-profit are benchmarked against the threshold value ζ. At the individual level, different newsvendors may have different heuristics $\alpha(\tilde{d})$, and hence different threshold value ζ. From the proof for Theorem 3, we know that a high value of $\alpha(\tilde{d})$ leads to a larger ζ, meaning that the newsvendor would consider a wide range of products as low-value. This is probably because the newsvendor with higher $\alpha(\tilde{d})$ has higher target profit and is more ambitious. As a result, she is more likely to consider a product as low-profit and over-order it.

We use Corollary 2 to further illustrate Theorem 3.

Corollary 2. Assume the random demand $\tilde{d}$ is uniformly distributed in $[\underline{d}, \bar{d}] \subset \Re^+$, and $\tau = (p - c) \times \alpha(\tilde{d})$, where $\alpha : \mathcal{B} \to \Re_+$ is a function of the random demand. Then there exists a threshold value

$$\zeta_U = 2 \times \frac{\alpha(\tilde{d}) - \underline{d}}{\bar{d} - \underline{d}} \tag{2.10}$$

such that if $\frac{p-c}{p} \geq \zeta_U$, we can find $y^* \leq y_N$ such that $y^* \in \arg\max_{y \geq 0} \rho_\tau\left(\tilde{v}(y)\right)$; and if $\frac{p-c}{p} \leq \zeta_U$, we can find $y^* \geq y_N$ such that $y^* \in \arg\max_{y \geq 0} \rho_\tau\left(\tilde{v}(y)\right)$.

Proof. Following the assumption of uniform demand and the proof of Theorem 3, we have

$$\begin{aligned}
\frac{r(\xi)}{p\xi} &= \frac{1}{\xi}\int_{\underline{d}}^{\underline{d}+\xi(\bar{d}-\underline{d})} \frac{x}{\bar{d}-\underline{d}} dx - \alpha(\tilde{d}) \\
&= \frac{1}{2\xi(\bar{d}-\underline{d})}\left((\underline{d}+\xi(\bar{d}-\underline{d}))^2 - \underline{d}^2\right) - \alpha(\tilde{d}) \\
&= \frac{2\underline{d}+\xi(\bar{d}-\underline{d})}{2} - \alpha(d).
\end{aligned}$$

Since p, $\xi > 0$, we have $r(\xi) \geq 0$, or over-ordering, if and only if $\xi \geq 2 \times \frac{\alpha(\tilde{d}-\underline{d})}{\bar{d}-\underline{d}} = \zeta_U$. $\square$

By (2.10) we can see that the threshold value ζ_U increases with $\alpha(d)$. To have a more concrete example, let $\underline{d} = 100$, $\bar{d} = 200$, $\alpha(\tilde{d}) = k\mathbb{E}[\tilde{d}] = 150k$ with k a constant that falls in the range of (0,1). Hence, the decision maker forms the target profit as $\tau = 150k(p-c)$. A start-up company may set a conservative target such that k has a low value of 80%, i.e., $\tau = 80\%(p - c)\mathbb{E}[\tilde{d}]$. From (2.10) we know that the threshold value is $\zeta_U = 0.4$. That is, the company would consider a product to be a high-profit one and under-order it if and only if the product has $\frac{p-c}{p} > 0.4$. In contrast, if the company is well-established and has higher tolerance for risk, it may set a more ambitious target such that $k = 90\%$, i.e., $\tau = 90\%(p-c)\mathbb{E}[\tilde{d}]$. Similarly we can get the threshold value $\zeta_U = 0.7$, which means a product is considered high-profit if and only if $\frac{p-c}{p} > 0.7$. Figure 2.1 provides a clear illustration on how the under-order and over-order regions change with the threshold values.

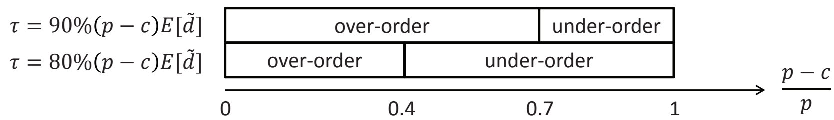

Fig. 2.1: High and low profit products for the two newsvendors.

Another stream of normative target setting is to study how should a firm set a target such that the decision made by the manager, who is driven by the target, will be the one optimizes the whole firm's objective. In accordance with this concept and assuming that the firm's objective being $CVaR_\eta$, $\eta \in [-1, 1]$, we have the following result.

Proposition 1. With the target value $\tau = \max_{y \geq 0} CVaR_\eta(\tilde{v}(y))$, we have

$$\arg\max_{y \geq 0} \rho_\tau(\tilde{v}(y)) = \arg\max_{y \geq 0} CVaR_\eta(\tilde{v}(y)).$$

Proof. It is obvious from the binary search procedure in finding the solution of the left hand side. □

According to Proposition 1, if the manager makes decision based on the target and using CSM criterion, the firm just needs to set the target level at the optimal $CVaR_\eta$ value. In that case, the manager's decision would be exactly as the same as the solution maximizing the $CVaR_\eta$ criterion.

2.2 Newsvendor with ESM

The second satisficing measure we consider for newsvendor decision is Entropic satisficing measure (ESM), which is focused on certainty equivalent achieving the target. Different from CSM, which considers only the worst/best case expectation, ESM captures all possible realizations of the random profit and hence represents decision makers' preference over the full scale. By assuming exponential utility function, we define the ESM as follows:

Definition 2. Given a target profit τ, the entropic satisficing measure (ESM), $\rho_\tau^E : \mathcal{B} \to [-\infty, \infty]$ is defined as:

$$\rho_\tau^E(\tilde{v}) = \begin{cases} \sup \{\eta : C_\eta(\tilde{v}) \geq \tau\}, & \text{if } feasible, \\ -\infty & \text{otherwise}, \end{cases} \tag{2.11}$$

where $C_\eta : \mathcal{B} \to \Re$ is defined as

$$C_\eta(\tilde{v}) = \begin{cases} -\frac{1}{\eta} \ln \mathbb{E}\left[\exp(-\eta\tilde{v})\right] & \text{if } \eta \neq 0, \\ \mathbb{E}[\tilde{v}] & \text{if } \eta = 0. \end{cases} \tag{2.12}$$

From Definition 2 we can see that, a random profit with higher ρ_τ^E attracts a greater subset of individuals who are willing to prefer the random profit over the target profit with certainty. In other words, an optimal newsvendor decision under the ESM framework is one such that this decision is favorable even to decision makers with very low risk tolerance level. This decision criterion can be especially useful for group decision making where each group member may have a different level of risk tolerance.

Interestingly, we show that for newsvendors under ESM, all the results in Section 2.1 still hold, which suggests that our model is robust with respect to the way decision makers perceive risks. Theorem 4 below summarizes the results.

Theorem 4. For newsvendors maximizing the Entropic satisficing measure, the following holds:

1. Assume that $\tau_1 \geq \tau_2$. Then there must exist $y_1 \geq y_2 \geq 0$ such that $y_i \in \arg\max_{y \geq 0} \rho^E_{\tau_i}(\tilde{v}(y))$, $i \in \{1, 2\}$.

2. If τ is greater than (equal to, or less than) $\mathbb{E}[\tilde{v}(y_N)]$, then we can find y^* greater than (equal to, or less than, respectively) y_N such that $y^* \in \arg\max_{y \geq 0} \rho^E_{\tau_i}(\tilde{v}(y))$.

3. If $\tau = \alpha \times (p - c)$, where α is a positive value depends on the knowledge of $\tilde{d}$ alone. Then $\exists \zeta \in [0, 1]$ such that if $\frac{p-c}{p} \geq \zeta$, $\exists y^* \leq y_N$ and $y^* \in \arg\max_{y \geq 0} \rho^E_{\tau_i}(\tilde{v}(y))$; and if $\frac{p-c}{p} \leq \zeta$, $\exists y^* \geq y_N$ and $y^* \in \arg\max_{y \geq 0} \rho^E_{\tau_i}(\tilde{v}(y))$.

4. With the target value $\tau = \max_{y \geq 0} C_\eta(\tilde{v}(y))$, we have

$$\arg\max_{y \geq 0} \rho^E_\tau(\tilde{v}(y)) = \arg\max_{y \geq 0} C_\eta(\tilde{v}(y)).$$

Proof. 1) By the same argument as we made in the proof of Theorem 1, we look at the existence of $y_i \in [\underline{d}, \bar{d}]$ to prove the result.

By definition, we can easily check that $\max_{y\geq 0} \rho^E_{\tau_1}(\tilde{v}(y)) \leq \max_{y\geq 0} \rho^E_{\tau_2}(\tilde{v}(y))$ since $\tau_1 \geq \tau_2$.

If $\max_{y\geq 0} \rho^E_{\tau_1}(\tilde{v}(y)) = -\infty$, we have $\rho^E_{\tau_1}(\tilde{v}(y)) = -\infty$ for all y. Choose $y_1 = \bar{d}$. For any $y_2 \in [\underline{d}, \bar{d}]$ such that $\rho_{\tau_2}(\tilde{v}(y_2)) = \max_{y\geq 0} \rho^E_{\tau_2}(\tilde{v}(y))$, we have $y_1 \geq y_2$.

If $\max_{y\geq 0} \rho^E_{\tau_2}(\tilde{v}(y)) = \infty$. Let $y^* \in [\underline{d}, \bar{d}]$ be an order quantity such that $\rho_{\tau_2}(\tilde{v}(y^*)) = \infty$. By Lemma 3, $\mathbb{P}(\tilde{v}(y^*) \geq \tau_2) = 1$, which implies $-cy^* + p\underline{d} \geq \tau_2$. Hence, we have $-c\underline{d} + p\underline{d} \geq \tau_2$, and $\rho_{\tau_2}(\tilde{v}(\underline{d})) = \infty$. Choose $y_2 = \underline{d}$. For any $y_1 \in [\underline{d}, \bar{d}]$ such that $\rho_{\tau_1}(\tilde{v}(y_1)) = \max_{y\geq 0} \rho^E_{\tau_1}(\tilde{v}(y))$, we have $y_1 \geq y_2$.

Now we consider the case of $-\infty < \max_{y\geq 0} \rho^E_{\tau_1}(\tilde{v}(y)) \leq \max_{y\geq 0} \rho^E_{\tau_2}(\tilde{v}(y)) < \infty$. Given τ, let η_τ be the maximal value of the ESM, i.e., $\eta_\tau = \max_{y\geq 0} \rho^E_{\tau}(\tilde{v}(y))$. Further, for the case of $\eta_\tau \in (-\infty, \infty)$, let y_τ be the maximizer of $C_{\eta_\tau}(\tilde{v}(y))$, i.e., $y_\tau = \arg\max_{y\geq 0} C_{\eta_\tau}(\tilde{v}(y))$.

By definition of ESM, we know y_τ is also a maximizer of ESM. Observe that

$$y_\tau = \arg\max_{y\geq 0} C_{\eta_\tau}(\tilde{v}(y)) = \begin{cases} \arg\max\limits_{y\geq 0} \mathbb{E}\left[-\exp\left(-\eta_\tau \tilde{v}(y)\right)\right], & \text{if } \eta_\tau \in (0, \infty); \\ \arg\max\limits_{y\geq 0} \mathbb{E}\left[\tilde{v}(y)\right], & \text{if } \eta_\tau = 0; \\ \arg\max\limits_{y\geq 0} \mathbb{E}\left[\exp\left(-\eta_\tau \tilde{v}(y)\right)\right], & \text{if } \eta_\tau \in (-\infty, 0). \end{cases}$$

Hence, we know y_τ is also a maximizer of the expectation of a utility function $u_\tau(\cdot)$, where $u_\tau(w) = -\text{sign}(\eta_\tau)\exp(-\eta_\tau w)$ if $\eta_\tau \neq 0$, and $u_\tau(w) = w$ if

$\eta_\tau = 0$.

Since $\tau_1 \geq \tau_2$, we know $-\infty < \eta_{\tau_1} \leq \eta_{\tau_2} < \infty$. If $0 < \eta_{\tau_1} \leq \eta_{\tau_2}$, $u_{\tau_1}(w) = -\exp(-\eta_{\tau_1} w)$ and $u_{\tau_2}(w) = -\exp(-\eta_{\tau_2} w)$. Both are nondecreasing concave functions, and there exists nondecreasing concave function $f(\cdot)$ such that $u_2(w) = f(u_1(w))$ for all w. Therefore, $y_{\tau_2} \leq y_{\tau_1} \leq y_N$ (Eeckhoudt et al. 1995).

If $\eta_{\tau_1} \leq \eta_{\tau_2} < 0$, $u_{\tau_1}(w) = \exp(-\eta_{\tau_1} w)$ and $u_{\tau_2}(w) = \exp(-\eta_{\tau_2} w)$. Both are nondecreasing convex functions, and there exists nondecreasing convex function $f(\cdot)$ such that $u_1(w) = f(u_2(w))$ for all w. Therefore, by Lemma 4, $y_{\tau_1} \geq y_{\tau_2} \geq y_N$.

If $\eta_{\tau_1} \leq 0 \leq \eta_{\tau_2}$, then $u_1(\cdot)$ is nondecreasing convex function while $u_2(\cdot)$ is nondecreasing concave function. So we get $y_{\tau_1} \geq y_N \geq y_{\tau_2}$.

2) While $\tau = \max_{y \geq 0} \mathbb{E}\left[\tilde{v}(y)\right]$, we know $\eta_\tau = 0$ and $y_\tau = y_N$. Others can be derived from part 1).

3) The proof is similar to that for Theorem 3.

4) It is obvious from the binary search procedure to find the solution for optimizing ESM. □

2.3 Computational Analysis

In this section we conduct a numerical study to compare the ordering decisions using our target based approaches (maximizing CSM and ESM) with those from maximizing expected profit, maximizing attainment probability, and the model of mean-variance analysis which is formulated by Choi et al.

(2008) as follows:

$$\begin{aligned} \min \quad & \mathbb{E}\left[(\tilde{v}(y) - \mathbb{E}[\tilde{v}(y)])^2\right] \\ \text{s.t.} \quad & \mathbb{E}\left[\tilde{v}(y)\right] \geq \tau \\ & y \geq 0. \end{aligned} \tag{2.13}$$

We let the demand follow a discrete uniform distribution over $\{1, 2, \ldots, 100\}$. Note that for given demand, a newsvendor instance can be characterized by the selling price and the critical-fractile. To capture a wide range of scenarios, we generate 50 newsvendor instances, where for each instance, the price is randomly sampled from the distribution $U[10, 20]$, and the critical fractile from $U[0.2, 0.8]$. For the models involving target profit, we set target $\tau = \varphi \max_{y \geq 0} \mathbb{E}[\tilde{v}(y)]$, where φ is a parameter describing the distance the target is away from the maximal expected profit. We have $\varphi = \{0.7, 0.8, 0.9, 1.1, 1.2, 1.3\}$. For each newsvendor instance, we first find the optimal ordering quantity for each model, and then compute the performances of the optimal decision with respect to different measures. Note that computational complexity is not an issue for newsvendor problem because it involves only one dimensional search if a closed form solution is unavailable. We do not take expected utility approach as a benchmark since it is not clear what utility function to use, and hence a fair comparison is hard to be achieved. Even with the popular exponential utility, $u(x) = 1 - \exp(-\eta x)$, the solution is sensitive to risk averse parameter η, which is an abstract concept and hard to be estimated. For example, with current demand distribution, if $p = 12$ and $c = 6$, the optimal ordering quantity will be 44 if $\eta = 0.001$, 20 if $\eta = 0.01$, 5 if $\eta = 0.1$, and 1 if $\eta = 1$.

Table 2.1 shows the average performance of all the 50 instances for each target level specified by φ. Note that when $\varphi > 1$, the mean-variance model in (2.13) is infeasible so we provide the solutions for this model only for $\varphi \leq 1$. Here Expected Loss (EL) is the expected value of the loss with respect to the target; Conditional Expected Loss (CEL) is the expected value of the loss conditioning on that there is a *strictly* positive loss. Value at Risk (VaR) is the threshold value that the newsvendor's loss does not exceed with a specified probability level. Note that for EL, CEL, and VaR, low values are desirable because they all measure losses.

We first observe that the optimal ordering decision for the attainment probability model is always the most conservative in the sense that compared to the optimal solutions for other models, it gives the lowest expected profit, standard deviation, expected loss, conditional expected loss, and VaR. This is because the optimal solution of the attainment probability model, $\tau/(p-c)$ (Lau 1980), is one such that the profit can never exceed τ. This is very conservative in nature. But for other models involving target, we require the expected profit (mean variance model), conditional expected profit (CSM model), or certainty equivalent (ESM model) to be no less than the target. Therefore, the optimal decisions of these models must not be as conservative as that in the attainment probability model. Note that although the attainment probability solution gives the lowest expected profit, the resulting lowest standard deviation and various types of losses makes it perhaps the best decision model to apply for newsvendors who just start their business

φ	Approach	Criterion						
		Expected profit	Standard deviation	Attainment probability	EL	CEL	VaR @ 95%	VaR @ 99%
0.7	Expected profit	223	246	67.42%	81	241	258	318
	Attainment probability	132	65	81.78%	27	137	47	106
	Mean variance	158	91	79.30%	33	151	80	139
	CSM	203	159	74.10%	51	190	158	217
	ESM	196	146	75.38%	47	182	140	199
0.8	Expected profit	223	246	65.88%	89	252	258	318
	Attainment probability	146	78	79.16%	35	156	64	123
	Mean variance	180	120	75.54%	47	179	113	173
	CSM	214	189	70.28%	68	218	191	250
	ESM	211	180	71.08%	64	211	179	238
0.9	Expected profit	223	246	64.28%	98	263	258	318
	Attainment probability	159	92	76.40%	45	176	81	141
	Mean variance	202	158	71.10%	65	212	157	216
	CSM	221	218	66.58%	86	247	225	284
	ESM	220	214	67.00%	84	244	219	279
1.1	Expected profit	223	246	61.20%	116	285	258	318
	Attainment probability	181	120	71.10%	67	215	114	174
	CSM	223	254	60.40%	120	291	270	329
	ESM	221	275	58.58%	131	304	297	356
1.2	Expected profit	223	246	59.70%	126	297	258	318
	Attainment probability	190	135	68.50%	80	235	131	190
	CSM	222	262	58.22%	134	308	280	340
	ESM	215	298	54.88%	155	333	331	390
1.3	Expected profit	223	246	58.02%	136	307	258	318
	Attainment probability	198	150	65.80%	94	254	148	207
	CSM	221	270	55.98%	149	324	290	350
	ESM	207	319	51.14%	181	359	362	421

EL=Expected Loss; CEL=Conditional Expected Loss; VaR=Value at Risk.

Tab. 2.1: Performance of Various Newsvendor Models

venture, because the most important concern for start-ups is survival, the consequence of losses can be unbearable for them.

In the low target scenarios ($\varphi < 1$), the mean-variance model gives the second most conservative solution. The reason is that the optimal solution for this model is such that the expected profit is equal to the target (Choi et al. 2008). However, the two satisficing models have optimal solutions with η greater than zero. A positive η under CSM suggests that $CVaR_\eta$ reaches the target, whereas under ESM this implies the certainty equivalent achieving the target. As such, the expected profit under both satisficing measures should be larger than the target.

Compared to the risk-neutral model that maximizes the expected profit, for CSM and ESM, when the target is smaller than $E[\tilde{v}(y_N)]$, the decrease in the target results in a decrease in the expected profit as well as the standard deviation and the loss-related performance measures (EL, CEL, and VaR). However, the decrease in the profit (maximum at 12%, 196 vs. 223) is much more mild than that in the standard deviation and other loss related measures (maximum at 46%, 228 vs. 118). On the other hand, when the target is larger than $E[\tilde{v}(y_N)]$, an increase in the target is associated with a decrease in the expected profit but an increase in the standard deviation and the loss-related measures. Similarly, the profit reduction is relatively mild (maximum at 7%, 207 vs. 223) while the loss-related measures increase quickly (maximum at 40%, 362 vs. 258). This suggests that compared to the risk-neutral model, CSM and ESM perform relatively well when the target is low. Further, compared to the attainment probability model, CSM and ESM have an advantage on the resulting expected profit, which is most significant when the target is small.

Finally, as the value of φ approaches to one (from both sides), we observe that the performance difference between different models decreases. The reason is that as the target moves close to the maximal expected profit, all target based models (except the attainment probability model) provide solutions close to the risk neutral one. As such, the difference becomes smaller.

To illustrate the effectiveness of our target-based approach, we compare the solutions from ESM approach under two different targets, and the solutions from the expected exponential utility approach. While the single-product newsvendor problem leads to random profit with a simple CDF curve (same shape as the CDF curve of demand distribution when underage, and jump to 1 when overage), we consider the extension to the multi-products newsvendor problem such that the CDF curve of random profit is more meaningful for comparison. In the multi-products newsvendor problem, the newsvendor can order product $i \in \{1, 2, \dots, n\}$ at unit cost of c_i, and sell it at unit price p_i . The random demand of product i is $\tilde{d}_i$, which is independent from demands of other products. With the criterion of ESM, the problem can be formulated as:

$$\max_{y_i \geq 0, i \in \{1,2,\dots,n\}} \rho_\tau^E \left(\sum_{i=1}^{n} \left(-c_i y_i + p_i \min\{y_i, \tilde{d}_i\} \right) \right). \tag{2.14}$$

By Definition 2, in order to solve the problem (2.14), we can perform a binary search on η, and for each η, we solve the following subproblem,

$$\max_{y_i \geq 0, i \in \{1,2,\dots,n\}} -\frac{1}{\eta} \ln \mathbb{E} \left[\exp \left(-\eta \sum_{i=1}^{n} \left(-c_i y_i + p_i \min\{y_i, \tilde{d}_i\} \right) \right) \right] \qquad \text{if } \eta \neq 0, \tag{2.15}$$

or $\max\limits_{y_i \geq 0, i \in \{1,2,\dots,n\}} \mathbb{E}[\sum\limits_{i=1}^{n}(-c_i y_i + p_i \min\{y_i, \tilde{d}_i\})]$ if $\eta = 0$. Since each demand is independent from others, the subproblem can be decomposed into n single-product newsvendor problems and hence can be solved. With the criterion of expected exponential utility, the problem is equivalent to (2.15).

In this numerical study, we let $n = 10$, $p_i = 12$ and $c_i = 6$ for all $i \in$

$\{1, 2, \ldots, n\}$. The random demand for each of the ten products is uniformly distributed in $[1, 100]$. Similar to previous computational study, we first solve solutions for all approaches, including ESM with $\tau = 1080$ and $\tau = 2020$, expected exponential utility with $\eta = 0.001$, $\eta = 0.01$, $\eta = 0.1$, and $\eta = 1$. Then we run simulation and compare different solutions on the CDF of the random profit. We can see Figure 2.2 for comparison. At first, we observe that the result from expected exponential utility is closely dependent on the selection of η, the abstractness of which limits the applicability. Comparing the two solutions from ESM, the solution from $\tau = 1080$ can achieve the target of 1080 better, in the sense that its loss respect to 1080 is less in both frequency and magnitude. On the other hand, the solution from $\tau = 2020$ can attain the target of 2020 better, since the solution from $\tau = 1080$ cannot beat the target at all. This comparison shows that the ESM approach provides solution beating the target well.

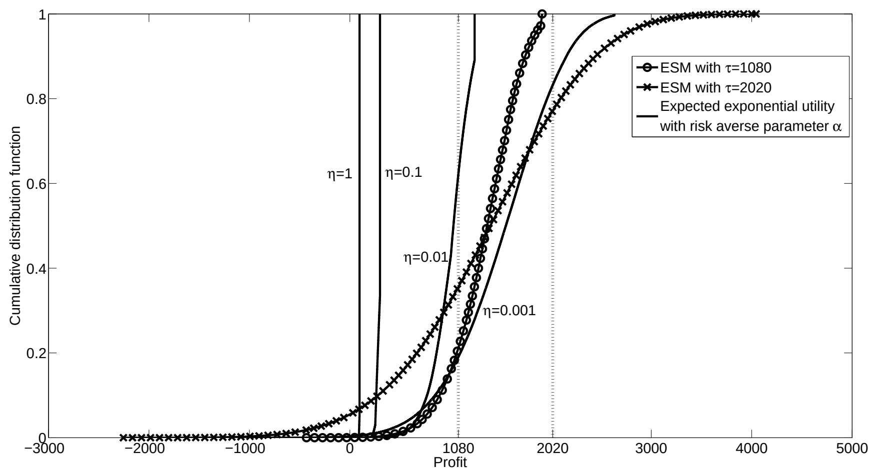

Fig. 2.2: CDF of random profits from solution of different approaches.

2.4 Conclusions

In this chapter, we incorporate the target profit in newsvendor decision making. By adopting the concept of satisficing measure (Brown et al. 2012), we study two measures: 1) CVaR satisficing measure (CSM) which measures the highest confidence level η which guarantees $CVaR_\eta$ achieving the target; and 2) Entropic satisficing measure (ESM) which measures the smallest risk tolerance level under which the certainty equivalent for an exponential utility function achieves the target.

We provide a method to easily find the optimal solution for the CSM framework, whereas for ESM the optimal decision can be numerically found. For both the two satisficing measures we find that i) the optimal order quantity increases with the target; ii) the newsvendor orders more than the risk neutral solution sometimes and order less other times, depending on the target level; and iii) if the target is proportional to the unit marginal profit and is also affected by only one other demand related factor, then the newsvendor over-orders the low-profit products and under-orders the high-profit products, which is consistent with the behavioral observation made by multiple studies. While Schweitzer and Cachon (2000) argue that "new techniques may be required to correctly optimize these systems", the consistency of our theoretical results and the existing behavioral observations suggest that our modelling framework may provides a potential direction in looking for the new techniques.

Most inventory models up to date focus on the absolute performance such as the expected profit. With ample evidence (e.g. Brown and Tang

2006) suggesting that managers are more concerned about achieving a target, our target-based framework opens a new direction for future research. We hope that our work would motivate increasing research interest along this avenue. We believe that it is worthwhile extending this framework to other operations management models to investigate how target affects decision making. It would also be interesting to study how people form their target profit in practice. Are there any heuristics that decision makers can follow or they simply use their intuition? Though past research has highlighted the role of target, few works have in fact analyzed the target formation. Although we proposed an intuitive target format in this chapter, we expect that more research remains to be done in this direction. Finally, while our results are consistent with the existing behavioral observations in laboratory experiments, we are in fact postulating that the subjects in these experiments are unconsciously setting targets. It is therefore worth further investigation to test this assumption.

2.5 Preliminary Lemmas 2 to 4

We now introduce Lemmas 2 to 4, which are used for subsequent proof of theorems.

Lemma 2. Given $\tilde{v} \in \mathcal{B}$ and $\tau \in \Re$, $\rho_\tau(\tilde{v}) = 1$ if and only $\mathbb{P}(\tilde{v} \geq \tau) = 1$.

Proof. If $\mathbb{P}(\tilde{v} \geq \tau) = 1$, $\forall \eta \in (0,1)$,

$$\begin{aligned} CVaR_{\eta}(\tilde{v}) &= \max_{a \in \Re} \left\{ a + \frac{1}{1-\eta} \mathbb{E}[\min\{\tilde{v} - a, 0\}] \right\} \\ &\geq \tau + \frac{1}{1-\eta} \mathbb{E}[\min\{\tilde{v} - \tau, 0\}] \\ &= \tau, \end{aligned}$$

where the second inequality holds for $\mathbb{P}(\tilde{v} \geq \tau) = 1$. Therefore, $\rho_{\tau}(\tilde{v}) = 1$.

If $\mathbb{P}(\tilde{v} \geq \tau) = 1 - 2\delta < 0$, where $\delta \in (0, 1/2]$, we have

$$CVaR_{1-\delta}(\tilde{v}) = \max_{a \in \Re} \left(a + \frac{1}{\delta} \mathbb{E}\left[\min\left\{\tilde{v} - a, 0\right\}\right] \right) = \max_{a \in \Re} \left\{ a + h(a) \right\},$$

with $h(a) = \frac{1}{\delta}\mathbb{E}\left[\min\left\{\tilde{v} - a, 0\right\}\right]$. For $a < \tau$, $a + h(a) \leq a < \tau$. For $a = \tau$, $a + h(a) = \tau + h(\tau) < \tau$, where the inequality holds because $\mathbb{P}(\tilde{v} \geq \tau) < 1$. For $a > \tau$,

$$\begin{aligned} a + h(a) &= a + \frac{1}{\delta} \left(\int_{\infty}^{\tau} (z - a) dF(z) + \int_{\tau}^{a} (z - a) dF(z) + \int_{a}^{\infty} 0 dF(z) \right) \\ &\leq a + \frac{1}{\delta} \left((\tau - a) \cdot 2\delta + 0 + 0 \right) \\ &= 2\tau - a \\ &< \tau. \end{aligned}$$

Therefore, we must have $CVaR_{1-\delta}(\tilde{v}) < \tau$, $\rho_{\tau}(\tilde{v}) < 1$. $\square$

Lemma 3. Given $\tilde{v} \in \mathcal{B}$ and $\tau \in \Re$, $\rho_{\tau}^{E}(\tilde{v}) = \infty$ if and only $\mathbb{P}(\tilde{v} \geq \tau) = 1$.

Proof. If $\mathbb{P}(\tilde{v} \geq \tau) = 1$, we can easily check that $C_{\eta}(\tilde{v}) \geq \tau$ for all η.

Therefore, $\rho_\tau^E(\tilde{v}) = \infty$.

If $\mathbb{P}(\tilde{v} \geq \tau) < 1$, $\exists \Delta < \tau$ such that $\mathbb{P}(\tilde{v} \leq \Delta) = \delta \in (0,1)$. Denote the upper bound of $\tilde{v}$ as $\bar{v}$.

$$\begin{aligned}
\lim_{\eta\to\infty} C_\eta(\tilde{v}) &= \lim_{\eta\to\infty} -\frac{1}{\eta}\ln \mathbb{E}\left[\exp(-\eta\tilde{v})\right] \\
&\leq \lim_{\eta\to\infty} -\frac{1}{\eta}\ln\left(\delta\exp(-\eta\Delta) + (1-\delta)\exp(-\eta\bar{v})\right) \\
&= \lim_{\eta\to\infty} -\frac{1}{\eta}\ln\left(\exp(-\eta\Delta)\left(\delta + (1-\delta)\exp(-\eta(\bar{v}-\Delta))\right)\right) \\
&= \Delta + \lim_{\eta\to\infty} -\frac{1}{\eta}\ln\left(\delta + (1-\delta)\exp(-\eta(\bar{v}-\Delta))\right) \\
&= \Delta + \lim_{\eta\to\infty} -\frac{1}{\eta}\ln(\delta) \\
&= \Delta \\
&< \tau.
\end{aligned}$$

Therefore, $\rho_\tau(\tilde{v}) < \infty$. □

Lemma 4. Assume u_1, u_2, and f are nondecreasing convex functions from $\Re$ to $\Re$, such that $u_1(w) = f(u_2(w))$, $\forall\, w \in \Re$. Then $\exists y_1 \geq y_2 \geq y_N$, such that $y_i \in \arg\max_{y\geq 0} \mathbb{E}\left[u_i(\tilde{v}(y))\right]$, $i \in \{1,2\}$.

Proof. For $i \in \{1,2\}$, denote $t_i(y) = \mathbb{E}\left[u_i(\tilde{v}(y))\right]$. Let y_2 be a maximizer of $t_2(\cdot)$ on $\Re_+$, i.e., $t_2(y_2) \geq t_2(y)$ for all $y \geq 0$. To prove the existence of

$y_1 \geq y_2$, it suffices to show $\forall y \leq y_2$, $t_1(y) \leq t_1(y_2)$. Observe

$$
\begin{aligned}
& t_1'(y) \\
= \ & \frac{\partial}{\partial y}\left(\int_0^y u_1(pd-cy)dF(d) + u_1(py-cy)(1-F(y))\right) \\
= \ & -c\int_0^y u_1'(pd-cy)dF(d) + (p-c)u_1'(py-cy)\,(1-F(y)) \\
= \ & -c\int_0^y f'(u_2(pd-cy))u_2'(pd-cy)dF(d) + (p-c)f'(u_2(py-cy))u_2'(py-cy)\,(1-F(y)) \\
\geq \ & s(y)t_2'(y),
\end{aligned}
$$

where $s(y) = f'(u_2(py-cy))$ is a nondecreasing function. Therefore, $\forall y \leq y_2$,

$$
t_1(y_2) - t_1(y) = \int_y^{y_2} t_1'(x)dx \geq \int_y^{y_2} s(x)t_2'(x)dx \geq \int_{m_1}^{y_2} s(x)t_2'(x)dx,
$$

where $m_1 = y$ if $t_2'(y) < 0$, and $m_1 = \max\{r \leq y_2 : \forall x \in [y,r], t_2'(x) \geq 0\}$ otherwise. Since y_2 is a maximizer of $t_2(\cdot)$, we can let

$$
\begin{aligned}
n_i &= \max\{r : \forall x \in [m_i, r], t_2'(x) \leq 0\}, \quad i = 1, 2, \ldots, N-1, \\
m_{i+1} &= \max\{r \leq y_2 : \forall x \in [n_i, r], t_2'(x) \geq 0\}, \quad i = 1, 2, \ldots, N-1,
\end{aligned}
$$

such that $y_2 = m_N$. Therefore,

$$
\begin{aligned}
& t_1(y_2) - t_1(y) \\
\geq\ & \int_{m_1}^{y_2} s(x)t_2'(x)dx \\
=\ & \sum_{i=1}^{N-1} \left(\int_{m_i}^{n_i} s(x)t_2'(x)dx + \int_{n_i}^{m_{i+1}} s(x)t_2'(x)dx \right) \\
\geq\ & \sum_{i=1}^{N-1} \left(s(n_i) \int_{m_i}^{m_{i+1}} t_2'(x)dx \right) \\
=\ & \sum_{i=1}^{N-3} \left(s(n_i) \int_{m_i}^{m_{i+1}} t_2'(x)dx \right) + s(n_{N-2}) \int_{m_{N-2}}^{m_{N-1}} t_2'(x)dx + s(n_{N-1}) \int_{m_{N-1}}^{m_N} t_2'(x)dx \\
\geq\ & \sum_{i=1}^{N-3} \left(s(n_i) \int_{m_i}^{m_{i+1}} t_2'(x)dx \right) + s(n_{N-2}) \int_{m_{N-2}}^{m_N} t_2'(x)dx \\
\geq\ & \sum_{i=1}^{N-4} \left(s(n_i) \int_{m_i}^{m_{i+1}} t_2'(x)dx \right) + s(n_{N-3}) \int_{m_{N-3}}^{m_N} t_2'(x)dx \\
& \vdots \\
\geq\ & s(n_1) \int_{m_1}^{m_N} t_2'(y)dy,
\end{aligned}
$$

where the second inequality holds since $s(x)$ is nondecreasing and $t_2'(x)$ is non-positive when $x \in [m_i, n_i]$ and non-negative when $x \in [n_i, m_{i+1}]$; the following inequalities hold since y_2 is maximizer of $t_2(\cdot)$ and hence $\int_{m_i}^{m_N} t_2'(x)dx \geq 0$.

Therefore, we get $t_1(y_2) - t_1(y) \geq s(n_1)(t_2(y_2) - t_2(m_1)) \geq 0$ and $t_1(y_2) \geq t_1(y)$, $\forall y \in [0, y_2]$. Therefore, we know that $\exists y_1 \geq y_2$ such that y_1 is the maximizer of $t_1(\cdot)$ on $\Re_+$.

Moreover, let $u_3(w) = w$. Then $u_3 : \Re \to \Re$ is an increasing convex function and $u_2(w) = u_2(u_3(w))$ for any w. Let $t_3(y) = \mathbb{E}\left[u_3(\tilde{v}(y))\right]$, and y_3 be the maximizer of $t_3(\cdot)$ on $\Re_+$. Then from the proof above we know

$y_2 \geq y_3 = y_N$, where the equality holds from the definition of y_N and y_3. $\square$

3. MANAGING OPERATIONAL AND FINANCING DECISIONS TO MEET CONSUMPTION TARGETS

Firms plan their operational decisions such as procurement, production, and inventory replenishment to generate cash flow for day-to-day expenditures (e.g., wages, dividend payments, and R&D costs) and more importantly, to grow the company. Due to multiple sources of randomness (e.g., demand volatility) embedded in the business environment, cash flow arising from any operational decision is risky and this complicates the decision making process. In principle, an analyst could construct a dynamic model to evaluate a decision that will reveal the corresponding cash flow profile, i.e., the probability distributions of the present values of the risky cash flow over time. The overarching challenges in a dynamic decision problem are to determine the appropriate decision criterion that translates the preference of the decision maker and also to find the "best" policy in which the criterion is optimized. Unfortunately, such a problem often suffers from the "curse of dimensionality" and its computational tractability depends critically on size of the underlying state-space, which may be aggravated by the choice of decision criterion. Even in the simplest setting where decision makers are risk neutral, finding the optimal policy that maximizes the expected net present value of the consumptions profile is #P-hard and possibly PSPACE-hard; see Dyer and Stougie (2006). Nevertheless, the optimal risk neutral policies of many important dynamic decision problems with smaller state space can be analyzed and solved via Bellman's (1957) dynamic programming.

Despite the technical attractiveness of risk neutral decision criteria, they neglect the risks involved in the operational cash flows and may not appeal to managers who are averse to potential losses along the planning horizon. They also ignore the fact that decision makers can be sensitive to the timing of the resolution of uncertainties. For example, suppose a firm will have a risky cash flow 10 years from now, knowing it today can be preferred than knowing it 10 years later because by having the knowledge of the cash flow today, managers can better plan (through borrowing and lending) corporate activities such as expansion and investment in new technology. Markowitz (1959), Matheson and Howard (1984) among others recognize this as the problem of "temporal risk." In order to resolve this problem, decision makers should be allowed to borrow or lend to smooth out *consumptions* over the planning horizon. In the corporate world, consumptions can refer to expenditures such as wages, dividend payments, and R&D costs. More details on "temporal risk problem" can be found in Smith (1998).

Recognizing the risks in the operational cash flow as well as the temporal risk, we study a firm's operational and financing decisions concurrently by modelling a finite horizon dynamic decision problem. We refer to *operational decisions* as those (e.g., production quantity and inventory planning) that would directly affect the operational cash flow in response to underlying uncertainties, and *financing decisions* as the amount of money the firm borrows or lends through financial markets. In every period except for the last, the total cash flow arising from the operational and financing decisions is the firm's consumption. To illustrate, if the firm needs to consume more than the operational cash flow, she'll borrow from the financial markets at a cost and if she consumes less, she will lend and earn the interest. The consumption in the final period is defined as the total wealth of the company in the consideration that a firm's ultimate goal is to grow the company. We define *consumptions profile* as the probability distributions of the present values of the risky consumptions over time.

One widely adopted criterion for evaluating the consumptions profile is expected utility, which captures decision maker's risk awareness and also has strong normative basis. Dynamic programming under the expected utility criterion has been proposed and studied by Howard and Matheson (1972), Porteus (1975) and Jaquette (1976) among others. Smith (1998) shows that under an additive-exponential utility function, joint operational and financing decisions can be made without overburdening the analysis. In the context of inventory management, Chen et al. (2007) show that this approach also preserves the structures of optimal joint inventory replenishment and pricing policies and hence, extends the result of Bouakiz and Sobel (1992).

However, as discussed in Chapter 1, it is of great interest to consider the effect of targets in this decision problem. To this end, we propose a new dynamic decision criterion, *Consumptions Profile Riskiness Index (CPRI)*, that evaluates the prospects of a consumptions profile in achieving consumptions targets over time. This decision criterion is especially of practical meaning, as an important aspect of managers' decision making is attainment of predetermined targets (of consumption levels, profits, or company stock prices), which is a direct reflection of their performance. Simon (1955), who has coined the term *satisfice*, elucidates that most firms' goals are not maximizing profits but attaining their target profits. From the interviews of executives in large corporations, Lanzillotti (1958) concludes that managers are primarily concerned about target returns on investment. Likewise, Mao (1970) also concludes from his empirical study that managers perceived *risk* as the "prospect of not meeting some target rate of return". From the normative perspective, research interests in target-oriented utility can be traced to Borch (1968) and have rekindled in recent years (Bordley and LiCalzi 2000, Bordley and Kirkwood 2004, Castagnoli and Calzi 1996, Tsetlin and Winkler 2007).

Our decision criterion CPRI is based on the extension of the riskiness index axiomatized by Aumann and Serrano (2008). Under this criterion, a joint operational and financing decision that returns the lowest CPRI is most preferred. We show that if there is no limit on the amount of borrowing and lending (*full financing*), all consumption targets, except the terminal one, are met with certainty. Moreover, the optimal policy of this criterion has the same control state as the optimal risk neutral policy and could be achieved with relatively modest computational effort. When financing is restricted, we show that for convex dynamic decision problems, the optimal policies correspond to those that optimize expected additive-exponential utilities. We also provide an algorithm to find the optimal policies.

Applying the CPRI decision framework to the joint inventory-pricing decision problem, we identify the optimal inventory and pricing policies for the case of with fixed ordering cost under full financing, and the policy structures for the case of zero fixed ordering cost under restricted financing. These results fill the void of inventory and inventory-pricing literature which has been mainly focused on risk neutral decision and expected utility to incorporate risk aversion. With our numerical studies for inventory control problems, we also report favorable computational results for using targets in regulating uncertain consumptions over time.

Structure of the chapter. Section 3.1 introduces the decision criterion CPRI. Section 3.2 discusses a general framework for joint operational and financing decisions under the CPRI. Optimal policies are provided for both the scenarios of full and restricted financing. Section 3.3 applies CPRI to a joint inventory and pricing decision problem and identifies the optimal inventory replenishment and pricing policies. Section 3.4 presents numerical results for inventory control problems. We conclude the chapter in Section 3.5.

3.1 Consumptions profile riskiness index (CPRI)

In this section, we propose a new decision criterion for evaluating the prospects of a consumptions profile in achieving consumptions targets over time. Although the joint probability of achieving targets is a natural candidate for a target-oriented decision criterion, Diecidue and Van de Ven (2008) argue against success probability as it tacitly assumes that the decision maker is indifferent to the magnitude of the losses when they occur. Our criterion is built upon the *riskiness index* recently axiomatized by Aumann and Serrano

(2008).

We first introduce some general notations used in the chapter. A vector such as $\boldsymbol{x}$ is represented as a boldface character and x^i denotes the ith component of the vector. In particular, $\mathbf{1}$ represents a vector of ones. Let $\mathcal{C}$ be a set of random variables on Ω in which $\tilde{c} \in \mathcal{C}$ denotes the present values of the uncertain consumptions that will be realized in future.

Definition 3. The riskiness index is a function, $\rho : \mathcal{C} \mapsto [0, \infty]$ defined as follows

$$\rho(\tilde{c}) = \min \left\{ \alpha \;\middle|\; \mathrm{C}_\alpha(\tilde{c}) \geq 0, \alpha > 0 \right\} \tag{3.1}$$

where $C_\alpha : \mathcal{C} \mapsto \Re$ is the certainty equivalent function defined as

$$\mathrm{C}_\alpha(\tilde{c}) = -\alpha \ln \mathbb{E}\left[\exp\left(-\tilde{c}/\alpha\right)\right].$$

By convention, we define $\inf \emptyset = \infty$.

Aumann and Serrano (2008) interpret the riskiness index as the reciprocal of the absolute risk aversion (ARA) of an individual with constant ARA who is indifferent between accepting and not accepting the uncertain consumptions. (While $\tilde{c}$ can refer to any random position such as investment return, we specifically call it consumption in the context of this chapter.) Uncertain consumptions with lower riskiness index appeal to a greater subset of individuals who are willing to accept the uncertain consumptions over nothing. The riskiness index has the same unit as the underlying consumptions and has the following properties (Aumann and Serrano 2008). It is *positively* *homogeneous*, i.e.,

$$\rho(k\tilde{c}) = k\rho(\tilde{c}) \qquad \forall k \geq 0,$$

and *subadditive*, i.e., for all $\tilde{c}_1, \tilde{c}_2 \in \mathcal{C}$,

$$\rho(\tilde{c}_1 + \tilde{c}_2) \leq \rho(\tilde{c}_1) + \rho(\tilde{c}_2).$$

The positively homogeneous property reflects upon the cardinal nature of riskiness such that $k\tilde{c}$ is k times as risky as $\tilde{c}$. Subadditivity implies that pooling of consumptions is less risky than the sum of the individual parts. These two properties imply that the riskiness index is also a convex function and hence consistent with convex preference. Furthermore, the riskiness index is monotone with second-order stochastic dominance and therefore it is a suitable decision making criterion for risk averse individuals.

Brown and Sim (2009) show that the riskiness index can be extended to a target-oriented decision criterion by evaluating the riskiness index of the consumption excesses over targets, $\tilde{c} - \tau$, where $\tau \in \Re$ is the target at present value. The target-oriented riskiness index embodies the property of *satisficing*, i.e., $\rho(\tilde{c} - \tau) = 0$ if and only if $\mathbb{P}(\tilde{c} \geq \tau) = 1$. Hence, similar to the probability measure, uncertain consumptions that can almost surely achieve the target will be most preferred and equally valued. However, unlike probability measure, this target-oriented criterion is diversification favoring and also rejects consumptions that fail to meet their targets in expectation, i.e., if $\mathbb{E}[\tilde{c}] < \tau$ then $\rho(\tilde{c} - \tau) = \infty$. We call this property *loss aversion awareness*, which captures the effect of loss aversion described by

Kahneman and Tversky (1979) that, the disutility of losses below the reference point or target is far larger than the value derived from the same size of gains. The loss aversion awareness property is reflected in Payne et al. (1980, 1981), which studies that managers tend to eradicate investment possibilities that underperform against their targets. Further, Brown et al. (2012) show that the target-oriented riskiness index can resolve several well-known behavioral experiments that contradict the expected utility theory. To illustrate this, we present the gambles of Allais' paradox in Table 3.1. Most subjects prefer Gamble A over Gamble B and Gamble C over Gamble D and this preference cannot be resolved by expected utility and also success probability, which can be perceived as a form of expected utility with step function. In contrast, this preference can be resolved via the target-oriented riskiness index. In particular, preferences for Gamble A over Gamble B and Gamble C over Gamble D are strict if $\tau \in (.023M, 0.25M]$. For $\tau \in (0.25M, 0.5M]$, Gamble A is strictly preferred over Gamble B, while both Gambles C and D are equally disliked since the expectation of the gambles are less than the target, τ[1].

Tab. 3.1: Gambles in Allais' paradox

Gamble A	Wins \$0.5M for sure.
Gamble B	Wins \$0M with 1% chance, \$2.5M with 10% chance, and \$0.5M with 89% chance.
Gamble C	Wins \$0M with 90% chance, and \$2.5M with 10% chance.
Gamble D	Wins \$0M with 89% chance, and \$0.5M with 11% chance.

[1] In Brown et al. (2012) aspirations measure, which incorporates risk seeking behavior and hence generalizes Brown and Sim (2009) sacrificing measures, the strict preference can be extended to $\tau \in (.023M, 0.5M]$.

Motivated by the normative and behavioral relevance of the target-oriented riskiness index, we generalize this approach and propose the new decision criterion CPRI for evaluating the prospect of a consumptions profile in achieving future targets. Let a vector of T random variables $\tilde{\boldsymbol{c}} = (\tilde{c}_1, \ldots, \tilde{c}_T)$ denote the decision maker's consumptions profile from period 1 to T, where $\tilde{c}_t \in \mathcal{C}$ is the present value of the uncertain consumption that will realize in period t. $\boldsymbol{\tau} = (\tau_1, \ldots, \tau_T)$ is a deterministic T-dimensional vector to represent the pre-determined consumptions targets from 1 to T, with τ_t as the present value of the consumption target in period t. (Note that if the interest rate compounded at every period is β, then the future consumption and target in period t would be $(1+\beta)^t \tilde{c}_t$ and $(1+\beta)^t \tau_t$, respectively.) Also let $\mathcal{C}^{\dim(\boldsymbol{\tau})}$ be the set of random vectors that have the same dimension as $\boldsymbol{\tau}$.

Definition 4. The consumptions profile riskiness index (CPRI) of a consumptions profile $\tilde{\boldsymbol{c}}$ with respect to the vector of targets $\boldsymbol{\tau}$ is a function, $\varphi_{\boldsymbol{\tau}} : \mathcal{C}^{\dim(\boldsymbol{\tau})} \mapsto [0, \infty]$ defined as follows:

$$\varphi_{\boldsymbol{\tau}}(\tilde{\boldsymbol{c}}) = \sum_{t=1}^{\dim(\boldsymbol{\tau})} \rho(\tilde{c}_t - \tau_t). \tag{3.2}$$

Essentially, the CPRI criterion is the sum of riskiness indices of the present values of the consumption excesses over targets. The present values are used to resolve the time value incompatibility of riskiness over different periods. The evaluation of the uncertain consumptions in each period is compartmentalized and hence, if there exist a period t with disfavoring uncertain consumptions such that $\rho(\tilde{c}_t - \tau_t) = \infty$, then the consumptions profile will be least favored, i.e., $\varphi_{\boldsymbol{\tau}}(\tilde{\boldsymbol{c}}) = \infty$. On the other hand, the most favored consumptions profile, i.e., $\varphi_{\boldsymbol{\tau}}(\tilde{\boldsymbol{c}}) = 0$ will require the stringent condition that all future consumptions attain their targets almost surely.

Proposition 2. The CPRI criterion, $\varphi_{\boldsymbol{\tau}} : \mathcal{C}^{\dim(\boldsymbol{\tau})} \mapsto [0, \infty]$ has the following properties:

1. *Satisficing*: $\varphi_{\boldsymbol{\tau}}(\tilde{\boldsymbol{c}}) = 0$ if and only if $\mathbb{P}(\tilde{\boldsymbol{c}} \geq \boldsymbol{\tau})$.

2. *Loss aversion awareness*: If there exists a time period t such that $\mathbb{E}[\tilde{c}_t] < \tau_t$ then $\varphi_{\boldsymbol{\tau}}(\tilde{\boldsymbol{c}}) = \infty$.

3. *Convexity*: For all $\tilde{\boldsymbol{c}}_1, \tilde{\boldsymbol{c}}_2 \in \mathcal{C}^{\dim(\boldsymbol{\tau})}$,

$$\varphi_{\boldsymbol{\tau}}(\lambda\tilde{\boldsymbol{c}}_1 + (1-\lambda)\tilde{\boldsymbol{c}}_2) \leq \lambda\varphi_{\boldsymbol{\tau}}(\tilde{\boldsymbol{c}}_1) + (1-\lambda)\varphi_{\boldsymbol{\tau}}(\tilde{\boldsymbol{c}}_2).$$

4. *Subadditivity*: For all $\tilde{\boldsymbol{c}} \in \mathcal{C}^{\dim(\boldsymbol{\tau})}$,

$$\varphi_{\boldsymbol{\tau}}(\tilde{\boldsymbol{c}}) \geq \varphi_{\mathbf{1}'\boldsymbol{\tau}}\left(\mathbf{1}'\tilde{\boldsymbol{c}}\right),$$

Proof. The first two properties are trivial to show. The last two properties follow directly from the homogeneity and subadditivity of riskiness index. □

We note that the convexity and subadditivity properties of the CPRI criterion have important ramifications that ensure tractable analysis in dynamic decision problems, which we will discuss in the next section.

3.2 Optimizing the CPRI criterion

We consider a firm making operational decisions (e.g., inventory planning, procurement) in the presence of uncertainties such as demand variability and supply volatility. The resulting cash flow from operational decisions are used for consumption as well as increasing firm's wealth (or value in other words). The firm has access to financial markets to borrow or lend at an interest rate of β in order to smooth out consumptions over the planning horizon T. With the objective of minimizing the CPRI, the firm needs to make operational as well as financing decisions in every period.

We have $\tilde{\boldsymbol{z}}_t : \Omega \mapsto \Re^n$, $t \in \{1, \ldots, T\}$ represent the vector of uncertainties in period t, which are independently distributed and resolved overtime. We further define the $n \times t$ vector $\boldsymbol{\zeta}_t = (\boldsymbol{z}_1, \ldots, \boldsymbol{z}_t)$, $t \in \{1, \ldots, T\}$ as the realizations of the uncertainties at the end of period t. We also define $\boldsymbol{\zeta}_0 = \{\}$. For convenience, we define the index sets, $\mathcal{T} = \{1, \ldots, T\}$ and $\mathcal{T}^- = \{1, \ldots, T-1\}$.

The sequence of events in any period t, $t \in \mathcal{T}^-$ is as follows: I) At the beginning of the period, the firm observes her state of wealth w_t and operational state of the system, $\boldsymbol{x}_t$, which is an element in $\Re^{s_t}$. II) The firm then administers an operational control $\boldsymbol{u}_t$, which takes values in a nonempty set $U_t(\boldsymbol{x}_t) \subseteq \Re^{v_t}$, i.e., $\boldsymbol{u}_t \in U_t(\boldsymbol{x}_t)$. III) Near the end of the period, the uncertainty $\tilde{\boldsymbol{z}}_t$ is resolved and takes a value of $\boldsymbol{z}_t$, which results in an operational cash flow of $r_t = g_t(\boldsymbol{x}_t, \boldsymbol{u}_t, \boldsymbol{z}_t)$. The operational state of the system is updated as $\boldsymbol{x}_{t+1} = \boldsymbol{f}_t(\boldsymbol{x}_t, \boldsymbol{u}_t, \boldsymbol{z}_t)$. IV) The firm then makes the financing decision to borrow b_t (or lend if $b_t < 0$). In the event that the financing is restricted, we note that it can be constrained by the firm's wealth level w_t and also depends on the system's updated operational state $\boldsymbol{x}_{t+1}$. So we assume $b_t \in F_t(\boldsymbol{x}_{t+1}, w_t)$. For the case of full financing, we have $F_t(\cdot,\cdot) = \Re$ and in the absence of financing, we have $F_t(\cdot,\cdot) = \{0\}$. V) Finally, the state of wealth is updated as $w_{t+1} = (1+\beta)w_t - b_t$.

The sequence of events in the terminal period T is the same as that for earlier periods except that there is no financing decision involved, i.e., $b_T = 0$. This is because we require all outstanding borrowing and interest earning to be settled by the end of period T so that the firm does not have outstanding balances. Given the different events in the terminal period and other periods, the present values of the consumptions over the horizon are given as follows. For $t \in \mathcal{T}^-$, $c_t = (r_t + b_t)/(1+\beta)^t$ and in the terminal period, $c_T = (r_T + (1+\beta)w_T)/(1+\beta)^T$.Taking a closer look at c_T, we can see that it represents the accumulated wealth gained from the operational cash flow subtracting the consumptions made in earlier periods.

Definition 5. An operations control state $\boldsymbol{s}_t^o \in \Re^{l_t^o}$, $t \in \mathcal{T}$ is non-anticipative if it is only influenced by the resolved uncertainty of $\boldsymbol{\zeta}_{t-1}$ and uninfluenced by future uncertainty $\boldsymbol{z}_t, \ldots, \tilde{\boldsymbol{z}}_T$. Similarly, a financing control state $\boldsymbol{s}_t^f \in \Re^{l_t^f}$, $t \in \mathcal{T}^-$ is non-anticipative if it is only influenced by the resolved uncertainty of $\boldsymbol{\zeta}_t$ and uninfluenced by future uncertainty $\tilde{\boldsymbol{z}}_{t+1}, \ldots, \tilde{\boldsymbol{z}}_T$.

We highlight that the *operational state* $\boldsymbol{x}_t$ refers to the state of the system in the current period t, and *operations (financing) control state* $\boldsymbol{s}_t^o$ $(\boldsymbol{s}_t^f)$ governs the operational (financing) policies in period t, which can contain more or less information than x_t.

Definition 6. An admissible operational policy is a sequence of T measurable functions given by $\boldsymbol{\Pi} = \{\boldsymbol{\pi}_1, \ldots, \boldsymbol{\pi}_T\}$ where $\boldsymbol{\pi}_t : \Re^{l_t^o} \mapsto \Re^{v_t}$ maps from a non-anticipative operations control state $\boldsymbol{s}_t^o$ into operational control $\boldsymbol{u}_t = \boldsymbol{\pi}_t(\boldsymbol{s}_t^o)$ and is such that $\boldsymbol{\pi}_t(\boldsymbol{s}_t^o) \in U_t(\boldsymbol{x}_t)$ for all possible states $\boldsymbol{s}_t^o$. Likewise, an admissible financing policy is a sequence of $T-1$ measurable functions given by $\boldsymbol{\Phi} = \{\phi_1, \ldots, \phi_{T-1}\}$ where $\phi_t : \Re^{l_t^f} \mapsto \Re$ maps from a non-anticipative financing control state $\boldsymbol{s}_t^f$ into financing decision $b_t = \phi_t(\boldsymbol{s}_t^f)$ and is such that $\phi_t(\boldsymbol{s}_t^f) \in F_t(\boldsymbol{x}_{t+1}, w_t)$ for all states $\boldsymbol{s}_t^f$.

We let $\mathcal{P}$ be the set of all admissible operational and financing policies. Starting with an initial operational state $\boldsymbol{x}_1$, an initial wealth w_1, and an admissible policy, $\boldsymbol{\Psi} = (\boldsymbol{\Pi}, \boldsymbol{\Phi}) \in \mathcal{P}$, the operational states $\tilde{\boldsymbol{x}}_t$ and the wealth of the firm $\tilde{w}_t$ are random variables with distributions defined through the following system equations:

$$\tilde{\boldsymbol{x}}_{t+1} = \boldsymbol{f}_t(\tilde{\boldsymbol{x}}_t, \boldsymbol{\pi}_t(\tilde{\boldsymbol{s}}_t^o), \tilde{\boldsymbol{z}}_t)$$

and

$$\tilde{w}_{t+1} = (1+\beta)\tilde{w}_t - \phi_t(\tilde{\boldsymbol{s}}_t^f)$$

for all $t \in \mathcal{T}^-$. The consumptions at the end of period t is a random variable given by

$$\tilde{c}_t(\boldsymbol{\Psi}) = \begin{cases} (g_t(\tilde{\boldsymbol{x}}_t, \boldsymbol{\pi}_t(\tilde{\boldsymbol{s}}_t^o), \tilde{\boldsymbol{z}}_t) + \phi_t(\tilde{\boldsymbol{s}}_t^f)/(1+\beta)^t & \text{if } t \in \mathcal{T}^-, \\ (g_T(\tilde{\boldsymbol{x}}_T, \boldsymbol{\pi}_T(\tilde{\boldsymbol{s}}_T^0), \tilde{\boldsymbol{z}}_T) + (1+\beta)\tilde{w}_T)/(1+\beta)^T & \text{if } t = T. \end{cases}$$

We define $\tilde{\boldsymbol{c}}(\boldsymbol{\Psi}) = (\tilde{c}_1(\boldsymbol{\Psi}), \ldots, \tilde{c}_T(\boldsymbol{\Psi}))$ to represent the consumptions profile as a function of the operational and financing policies, $\boldsymbol{\Psi} \in \mathcal{P}$.

Definition 7. A history dependent policy is an admissible policy in which the control states are the history of resolved uncertainties, i.e.,

$$\begin{aligned} \boldsymbol{s}_t^o &= \boldsymbol{\zeta}_{t-1} \quad t \in \mathcal{T}, \\ \boldsymbol{s}_t^f &= \boldsymbol{\zeta}_t \qquad t \in \mathcal{T}^-. \end{aligned}$$

We define $\mathcal{P}_H$ as the set of all admissible history dependent operational and financing policies.

Note that $\boldsymbol{s}_{t+1}^o = \boldsymbol{s}_t^f$. This is because by the sequence of events described earlier, there is one time period lag between the operations control states and the financing control states.

In classical dynamic programming in which expected reward is maximized, the optimal policy is usually dependent on system states instead of being superfluously dependent on its history. However, depending on the decision criterion, the system states may not necessarily be sufficient to describe the optimal policy. Nevertheless, although there exists an history dependent policy that is also optimal, such a policy can lead to the “curse of dimensionality" and are computationally intractable to implement in practice. In subsequent sections, we will show that in some interesting cases, optimal control policies may also be concisely dependent on system states, which has important ramifications on computations.

The consumptions profile that minimizes the target-oriented CPRI criterion and the associated optimal policy can be obtained by solving the following optimization problem:

$$\begin{aligned} \varphi^* = \min \quad & \varphi_{\boldsymbol{\tau}}\left(\tilde{\boldsymbol{c}}(\boldsymbol{\Psi})\right) \\ \text{s.t.} \quad & \boldsymbol{\Psi} \in \mathcal{P}_H, \end{aligned} \tag{3.3}$$

where $\varphi_{\boldsymbol{\tau}}$ is defined as in (3.2).

By the expressions of $\tilde{c}_t$ introduced in the earlier part of Section 3.2, we know that $\tilde{c}_t$, $t \in \mathcal{T}^-$ refers to the consumption and $\tilde{c}_T$ refers to the firm's wealth increase after consumptions in the previous periods. The target-oriented CPRI framework is thus consistent with such a scenario that when managers make decisions for a planning horizon, while meeting day-to-day corporate consumptions such as labor cost is important, at the end of the horizon, what's more crucial is whether achieving a pre-determined target profit, which often plays a significant role in managers' performance evaluation.

In this model, we say a policy $\boldsymbol{\Psi} \in \mathcal{P}_H$ is feasible if $\varphi_{\boldsymbol{\tau}}\left(\tilde{\boldsymbol{c}}(\boldsymbol{\Psi})\right) < \infty$. Hence, we assume that the decision maker can aptly set her targets, $\boldsymbol{\tau}$ so that $\varphi^* \in (0, \infty)$.

There are several criticisms of Model (3.3) which are related to the setting of targets. If there exist some policies such that their corresponding consumptions profiles achieve the targets almost surely, then $\varphi^* = 0$ and the CPRI criterion cannot distinguish among any of these profiles. In the other extreme, if there does not exist a policy that yields a consumptions profile with finite CPRI, then $\varphi^* = \infty$ and this framework would fail to obtain a feasible policy. Nevertheless, it is not unreasonable to assume that targets are set based on the perceived economic outlook and one could argue that the decision maker is not overly pessimistic or optimistic in setting targets. Simon (1955) provides an example of selling a house and the agent's target is determined after she learns about the climate of the housing market. Similarly, we may also argue that one may conjure her targets after examining the consumptions profiles of some policies such as the risk neutral optimal policy.

Another related criticism is the lack of time consistency such that an optimal policy perceived in one time period may not be perceived as optimal in another. While the decision maker may change her targets as uncertainty resolves over time, time inconsistency may occur even when her targets remain fixated. For instance, it is plausible that when the economic outlook is bad, the targets imposed at earlier periods may no longer yield a feasible policy unless the decision maker is willing to lower her targets. Time inconsistency also occurs in dynamic decision problems with non-exponential discounting factors. Nevertheless, while time consistency is a desirable feature in dynamic decision making, it is violated in behavioral experiments even in the absence of uncertainty; see for instance Thaler (1981), Frederick et al. (2002), and Loch and Wu (2007). Loewenstein (1988) shows experimental-ly that shifts of reference points, or targets in our language, could better account for behavioral time consistency than discounted utility models. In some situations, we may circumvent the issue of time inconsistency by simply adhering to the original policy announced in the first period, which may be applicable when there are heavy penalties against deviations from original plans. In other situations, it may be reasonable to implement a rolling (or folding) horizon approach to dynamic decision making in which only the "here-and-now" solution is obtained and implemented in every period without the need to announce future "wait-and-see" policy. This is achieved by solving the optimization problem based on a fresh set of targets and using the latest information available whenever we need to make and implement the decisions.

Despite these valid criticisms, we will next show that in some interesting cases, optimizing over consumptions profiles under the CPRI criterion can be made almost as easy as solving the underlying dynamic programming model. We will also show numerically on an inventory control problem that by using this approach we can better regulate consumptions over time.

3.2.1 Optimal policy under full financing

We first analyze the case in which the decision maker has unrestricted access to borrowing and lending to finance consumptions in periods $t \in \mathcal{T}^-$, which is similar to the assumptions made in Smith (1998) and Chen et al. (2007) to obtain tractable analysis. In the absence of financing restrictions, we will show that there exists an optimal financing policy in which the consumptions at periods $t \in \mathcal{T}^-$ are exactly at their targets. We call this a *financing-at-target (FAT)* policy.

Definition 8. Given an admissible operational policy $\mathbf{\Pi}$, the financing-at-target (FAT) policy is a financing policy $\mathbf{\Phi} = \{\phi_1, \ldots, \phi_{T-1}\}$, $\phi_t : \Re \mapsto \Re$ such that for all $t \in \mathcal{T}^-$,

$$\phi_t(r_t) = (1+\beta)^t \tau_t - r_t,$$

where $r_t = g_t(\boldsymbol{x}_t, \boldsymbol{u}_t, \boldsymbol{z}_t)$ is the realized operational cash flow in period t under policy $\mathbf{\Pi}$.

Theorem 5. Under full financing, there exists an optimal FAT policy that minimizes the CPRI criterion.

Proof. Let $(\mathbf{\Pi}^*, \mathbf{\Phi}^*) \in \mathcal{P}_H$ be an optimal admissible history dependent policy to Model (3.3) in which $\mathbf{\Phi}^*$ may not be a FAT policy. For given operational policy $\mathbf{\Pi}^*$, we show that the corresponding FAT policy, $\hat{\mathbf{\Phi}}$ would not yield a consumptions profile with worse CPRI. The consumptions under the FAT policy is

$$\tilde{c}_t = \begin{cases} \tau_t & \text{for } t \in \mathcal{T}^-, \\ \displaystyle\sum_{k \in \mathcal{T}} \frac{\tilde{r}_k}{(1+\beta)^k} + w_1 - \sum_{t \in \mathcal{T}^-} \tau_k & \text{for } t = T. \end{cases}$$

where $\tilde{r}_t$, $t \in \mathcal{T}$ is the uncertain operational cash flows under the operational policy $\mathbf{\Pi}^*$. Let $\tilde{\boldsymbol{c}}^* = (\tilde{c}_1^*, \ldots, \tilde{c}_T^*)$ denote the consumptions profile under the optimal policy $(\boldsymbol{\Pi}^*, \boldsymbol{\Psi}^*)$ and $\tilde{b}_t$, $t \in \mathcal{T}^-$ denotes the corresponding financing cash flows. Therefore,

$$\tilde{c}_t^* = \begin{cases} \dfrac{\tilde{r}_t + \tilde{b}_t}{(1+\beta)^t} & \text{for } t \in \mathcal{T}^-, \\ w_1 + \dfrac{\tilde{r}_T}{(1+\beta)^T} - \displaystyle\sum_{k \in \mathcal{T}^-} \dfrac{\tilde{b}_k}{(1+\beta)^k} & \text{for } t = T. \end{cases}$$

Hence, by subadditivity property of the CPRI criterion, we have

$$\begin{aligned} \varphi_{\boldsymbol{\tau}}\left(\tilde{\boldsymbol{c}}^*\right) &\geq \varphi_{\mathbf{1}'\boldsymbol{\tau}}\left(\mathbf{1}'\tilde{\boldsymbol{c}}^*\right) \\ &= \varphi_{\mathbf{1}'\boldsymbol{\tau}}\left(\sum_{t \in \mathcal{T}^-}\left(\frac{\tilde{r}_t + \tilde{b}_t}{(1+\beta)^t}\right) + w_1 + \frac{\tilde{r}_T}{(1+\beta)^T} - \sum_{t \in \mathcal{T}^-}\frac{\tilde{b}_t}{(1+\beta)^t}\right) \\ &= \rho\left(\sum_{t \in \mathcal{T}}\frac{\tilde{r}_t}{(1+\beta)^t} + w_1 - \sum_{t \in \mathcal{T}}\tau_t\right) \\ &= \sum_{t \in \mathcal{T}^-}\rho(\tilde{c}_t - \tau_t) + \rho\left(\tilde{c}_T - \tau_T\right) \\ &= \varphi_{\boldsymbol{\tau}}\left(\tilde{\boldsymbol{c}}\right). \end{aligned}$$

□

The optimality of the FAT policy implies that as long as Model (3.3) is feasible, one can always attain the desired consumption targets through financing, with the exception of the last target. Hence, we may perfectly regulate consumptions in periods $t \in \mathcal{T}^-$ by minimizing the CPRI and in doing so, relegate consumption uncertainty to the last period. We observe that the corresponding CPRI of the consumptions profile under the FAT policy can be expressed as follows:

$$\rho\left(\sum_{t\in\mathcal{T}}\frac{\tilde{r}_t}{(1+\beta)^t}+w_1-\sum_{t\in\mathcal{T}}\tau_t\right)=\inf\left\{\alpha>0 \;\middle|\; \mathrm{C}_\alpha\left(\sum_{t\in\mathcal{T}}\frac{\tilde{r}_t}{(1+\beta)^t}\right)\geq\sum_{t\in\mathcal{T}}\tau_t-w_1\right\},$$

where $\tilde{r}_t$, $t\in\mathcal{T}$ are the uncertain operational cash flows. Therefore, we can formulate the optimization problem to minimize CPRI as follows:

$$\begin{aligned}\varphi^* \;=\; & \min\alpha \\ \text{s.t.}\; & \max_{\mathbf{\Pi}\in\mathcal{Q}}\ \mathrm{C}_\alpha\left(\sum_{t\in\mathcal{T}}\frac{\tilde{r}_t(\mathbf{\Pi})}{(1+\beta)^t}\right)\geq\sum_{t\in\mathcal{T}}\tau_t-w_1 \\ & \alpha\geq 0,\end{aligned} \tag{3.4}$$

where $\mathcal{Q}$ is the set of all admissible operational policies and $\tilde{r}_t(\mathbf{\Pi})$ denotes the uncertain operational cash flows under policy $\mathbf{\Pi}\in\mathcal{Q}$. We next present the optimal operational policy.

Theorem 6. Under full financing, there exists an optimal operational state dependent policy that can be obtained by solving the dynamic programming given by

$$\boldsymbol{\pi}_t(\boldsymbol{x}_t)=\begin{cases}\arg\max\limits_{\boldsymbol{u}_T\in U_T(\boldsymbol{x}_T)}\mathrm{C}_{\varphi^*}\left(\dfrac{g_T(\boldsymbol{x}_T,\boldsymbol{u}_T,\tilde{\boldsymbol{z}}_T)}{(1+\beta)^T}\right) & \text{for } t=T,\\ \arg\max\limits_{\boldsymbol{u}_t\in U_t(\boldsymbol{x}_t)}\mathrm{C}_{\varphi^*}\left(\dfrac{g_t(\boldsymbol{x}_t,\boldsymbol{u}_t,\tilde{\boldsymbol{z}}_t)}{(1+\beta)^t}+L_{t+1}^{\varphi^*}(f_t(\boldsymbol{x}_t,\boldsymbol{u}_t,\tilde{\boldsymbol{z}}_t))\right) & \text{for } t\in\mathcal{T}^-,\end{cases}$$

where

$$L_t^\alpha(\boldsymbol{x}_t) = \begin{cases} \max\limits_{\boldsymbol{u}_T \in U_T(\boldsymbol{x}_T)} \mathrm{C}_\alpha \left(\dfrac{g_T(\boldsymbol{x}_T, \boldsymbol{u}_T, \tilde{\boldsymbol{z}}_T)}{(1+\beta)^T} \right) & \text{for } t = T, \\ \max\limits_{\boldsymbol{u}_t \in U_t(\boldsymbol{x}_t)} \mathrm{C}_\alpha \left(\dfrac{g_t(\boldsymbol{x}_t, \boldsymbol{u}_t, \tilde{\boldsymbol{z}}_t)}{(1+\beta)^t} + L_{t+1}^\alpha(f_t(\boldsymbol{x}_t, \boldsymbol{u}_t, \tilde{\boldsymbol{z}}_t)) \right) & \text{for } t \in \mathcal{T}^-, \end{cases}$$

defined for $\alpha > 0$ and

$$\varphi^* = \min \left\{ \alpha > 0 \;\middle|\; L_1^\alpha(\boldsymbol{x}_1) \geq \sum_{t \in \mathcal{T}} \tau_t - w_1 \right\}.$$

Moreover, $L_1^\alpha(\boldsymbol{x}_1)$ is an nondecreasing function of α and hence, φ^* can be found by a standard binary search on α.

Proof. In solving Problem (3.4), we observe the certainty equivalent function C_α is nondecreasing in $\alpha \in (0, \infty)$ and hence, we can obtain φ^* by standard binary search on α. For a given parameter, α, the subproblem to maximize certainty equivalent of total operational cash flows under exponential utility function can be solved by modifying standard dynamic programming (see Bertsekas 2005, Volumne 1, p. 53-54) so that

$$L_1^\alpha(\boldsymbol{x}_1) = \max_{\boldsymbol{\Pi} \in \mathcal{Q}} \mathrm{C}_\alpha \left(\sum_{t \in \mathcal{T}} \frac{\tilde{r}_t(\boldsymbol{\Pi})}{(1+\beta)^t} \right).$$

Since this is a standard approach, we omit the proof for brevity. $\square$

3.2.2 Optimal policy for convex dynamic decision problems

We now consider the general case when financing is restricted. Since a FAT policy may not necessarily be admissible, it would not be always possible to obtain the optimal policy by solving a small collection of dynamic programming problems as we have done in Theorem 6. Nevertheless, we will focus on a special class of convex dynamic decision problems in which the structure of the optimal policies under the CPRI criterion can be analyzed. Analogous to a convex maximization problem, the feasible policies of a convex dynamic decision problem is convex and the consumptions are concave functions with respect to the policies. The first step is to ensure that the feasible set of the optimization problem is closed, which is necessary for our results to hold. Hence, we analyze the policy of an ϵ-closure of Model (3.3) defined as follows

$$\begin{aligned} \varphi_\epsilon^* = \min \quad & \textstyle\sum_{t\in\mathcal{T}} \alpha_t \\ \text{s.t.} \quad & \mathrm{C}_{\alpha_t}(\tilde{c}_t(\boldsymbol{\Psi})) \geq \tau_t, \quad t \in \mathcal{T} \\ & \boldsymbol{\Psi} \in \mathcal{P}_H \\ & \boldsymbol{\alpha} \geq \mathbf{1}\epsilon, \end{aligned} \tag{3.5}$$

where $\epsilon > 0$ is a small number. Since $\mathrm{C}_\alpha(\cdot)$ is nondecreasing in α, we can establish that

$$\varphi^* \leq \varphi_\epsilon^* \leq \varphi^* + T\epsilon,$$

and hence, the optimal policy of Model (3.5) can be made arbitrarily close to that of Model (3.3). Therefore, with an abuse of terminology, we refer to an optimal policy of Model (3.5) as one that also minimizes the CPRI criterion.

From here forward, we assume finite discrete distributions, i.e., $\Omega = \{\omega_1, \ldots, \omega_K\}$ and that $\mathbb{P}\{\omega_k\} > 0$. While this assumption aims to simplify the analysis, it is not practically limiting in most applications of dynamic decision problems. Under the assumption of finite sample space, at any point in time, there are only finitely many possible histories or states that will influence control decisions. Therefore, the history dependent policies can be perceived as vectors representing concatenation of controls corresponding to all the possible control states. Hence, Model (3.5) can be expressed as a finite dimensional optimization problem. Despite the case, there could be exponentially large number of decision variables and computationally prohibitive to solve Model (3.5) directly as a mathematical optimization problem. Instead, we propose to address the problem by solving a sequence of dynamic optimization problems, which may enable us to exploit the structures of their optimal policies for efficient computations. We make further assumptions on the problem.

Assumption 1. The set of admissible policies $\mathcal{P}_H$ is closed, bounded, convex, i.e. for all $\boldsymbol{\Psi}^1, \boldsymbol{\Psi}^2 \in \mathcal{P}_H$,

$$\lambda\boldsymbol{\Psi}^1 + (1-\lambda)\boldsymbol{\Psi}^2 \in \mathcal{P}_H \qquad \forall\lambda \in [0,1]$$

and strictly feasible, i.e. $\exists\boldsymbol{\Psi} \in int\mathcal{P}_H$, where $int\mathcal{P}_H$ refers to the interior of $\mathcal{P}_H$, such that $\varphi(\tilde{\boldsymbol{c}}(\boldsymbol{\Psi})) \in [0,\infty)$. The consumptions are concave with respect to the policy, i.e, for all $t \in \mathcal{T}$,

$$\tilde{c}_t(\lambda\mathbf{\Psi}^1 + (1-\lambda)\mathbf{\Psi}^2, \omega) \geq \lambda\tilde{c}_t(\mathbf{\Psi}^1, \omega) + (1-\lambda)\tilde{c}_t(\mathbf{\Psi}^2, \omega) \qquad \forall\lambda \in [0,1], \omega \in \Omega.$$

Theorem 7. Under Assumption 1, the optimal policy under the CPRI criterion is one that maximizes the expected value of an additive-exponential utility function as follows:

$$\begin{aligned} \max \quad & \mathbb{E}\left[\sum_{t=1}^{T} -\delta_t \exp\left(-\frac{\tilde{c}_t\left(\mathbf{\Psi}\right)}{\alpha_t}\right)\right] \\ \text{s.t.} \quad & \mathbf{\Psi} \in \mathcal{P}_H, \end{aligned} \tag{3.6}$$

where $\boldsymbol{\alpha} > \mathbf{0}$ and $\boldsymbol{\delta} \geq \mathbf{0}$ take some specific values.

Proof. Note for any $\tilde{c} \in \mathcal{C}$, $\alpha > 0$, $\mathrm{C}_\alpha(\tilde{\mathrm{c}}) \geq 0$ if and only if $\alpha\mathbb{E}[\exp(-\tilde{c}/\alpha)] - \alpha \leq 0$. We can therefore formulate Model (3.5) equivalently as follows:

$$\begin{aligned} \min \quad & \textstyle\sum_{t\in\mathcal{T}} \alpha_t \\ \text{s.t.} \quad & \alpha_t \mathbb{E}\left[\exp\left(-(\tilde{c}_t(\mathbf{\Psi}) - \tau_t)/\alpha_t\right)\right] \leq \alpha_t, \quad t \in \mathcal{T} \\ & \mathbf{\Psi} \in \mathcal{P}_H, \\ & \boldsymbol{\alpha} \geq \mathbf{1}\epsilon. \end{aligned} \tag{3.7}$$

We claim that the function $\alpha\mathbb{E}\left[\exp\left(-(\tilde{c}_t(\mathbf{\Psi}) - \tau)/\alpha\right)\right]$ is jointly convex in $(\alpha, \mathbf{\Psi})$, $\alpha > 0$. Indeed, given $\alpha^1, \alpha^2 > 0$, $\mathbf{\Psi}^1, \mathbf{\Psi}^2 \in \mathcal{P}_H$, let $\alpha^\lambda = \lambda\alpha^1 + (1 -$

$\lambda)\alpha^2$ and $\mathbf{\Psi}^\lambda = \lambda\mathbf{\Psi}^1 + (1-\lambda)\mathbf{\Psi}^2$ for $\lambda \in (0,1)$. Observe that

$$\begin{aligned}
& \alpha^\lambda \mathbb{E}\left[\exp\left(-(\tilde{c}_t(\mathbf{\Psi}^\lambda) - \tau)/\alpha^\lambda\right)\right] \\
\leq\ & \alpha^\lambda \mathbb{E}\left[\exp\left(-(\lambda\tilde{c}_t(\mathbf{\Psi}^1) + (1-\lambda)\tilde{c}_t(\mathbf{\Psi}^2) - \tau)/\alpha^\lambda\right)\right] \\
=\ & \alpha^\lambda \mathbb{E}\left[\exp\left(-\tfrac{\lambda\alpha^1}{\lambda\alpha^1+(1-\lambda)\alpha^2}(\tilde{c}_t\left(\mathbf{\Psi}^2\right) - \tau)/\alpha^1 - \tfrac{(1-\lambda)\alpha^2}{\lambda\alpha^1+(1-\lambda)\alpha^2}(\tilde{c}_t\left(\mathbf{\Psi}^2\right) - \tau)/\alpha^2\right)\right] \\
\leq\ & \lambda\alpha^1 \mathbb{E}\left[\exp\left(-(\tilde{c}_t\left(\mathbf{\Psi}^2\right) - \tau)/\alpha^1\right)\right] + (1-\lambda)\alpha^2 \mathbb{E}\left[\exp\left(-(\tilde{c}_t\left(\mathbf{\Psi}^2\right) - \tau)/\alpha^2\right)\right]
\end{aligned}$$

where the first inequality holds for the convexity of the dynamic decision problem, the second inequality follows from the convexity of the exponential function. Hence, Problem (3.7) is a convex optimization problem with finite number of decision variables and it is strictly feasible by assumption. We let this be the primal problem and the Lagrange dual problem follows:

$$\begin{array}{ll} \max & g(\boldsymbol{\lambda}) \\ \text{s.t.} & \boldsymbol{\lambda} \geq \mathbf{0}, \end{array} \tag{3.8}$$

where

$$g(\boldsymbol{\lambda}) = \min_{\boldsymbol{\alpha}\geq \mathbf{1}\epsilon,\ \mathbf{\Psi}\in\mathcal{P}_H} L(\boldsymbol{\alpha}, \mathbf{\Psi}, \boldsymbol{\lambda}),$$

$$L(\boldsymbol{\alpha}, \mathbf{\Psi}, \boldsymbol{\lambda}) = \sum_{t\in\mathcal{T}} \left(\alpha_t + \lambda_t\left(\alpha_t \mathbb{E}\left[\exp\left(-(\tilde{c}_t(\mathbf{\Psi}) - \tau_t)/\alpha_t\right)\right] - \alpha_t\right)\right).$$

Since the primal problem has finite objective, is convex and strictly feasible, strong duality holds and the dual variables $\boldsymbol{\lambda}$ are attainable (See for instance Boyd and Vandenberghe 2004, section 5.2.3). Let $(\boldsymbol{\alpha}^*, \mathbf{\Psi}^*)$ be any optimal solution to the primal problem (3.7), and $\boldsymbol{\lambda}^*$ be any optimal solution to the dual problem (3.8). We have

$$L(\boldsymbol{\alpha}^*, \boldsymbol{\Psi}^*, \boldsymbol{\lambda}^*) = \sum_{t\in\mathcal{T}} \left(\alpha_t^* + \lambda_t^* \left(\alpha_t^* \mathbb{E}\left[\exp\left(-(\tilde{c}_t(\boldsymbol{\Psi}^*) - \tau_t)/\alpha_t^*\right)\right] - \alpha_t^*\right)\right) = \sum_{t\in\mathcal{T}} \alpha_t^* = g(\boldsymbol{\lambda}^*).$$

Therefore, $L(\boldsymbol{\alpha}^*, \boldsymbol{\Psi}^*, \boldsymbol{\lambda}^*) = \min_{\boldsymbol{\Psi}\in\mathcal{P}_H} L(\boldsymbol{\alpha}^*, \boldsymbol{\Psi}, \boldsymbol{\lambda}^*)$, and $\boldsymbol{\Psi}^*$ is an optimal solution to the problem given by

$$\max_{\boldsymbol{\Psi}\in\mathcal{P}_H} \mathbb{E}\left[\sum_{t\in\mathcal{T}} -\delta_t^* \exp(-\tilde{c}_t(\boldsymbol{\Psi})/\alpha_t^*)\right],$$

where $\delta_t^* = \lambda_t^* \alpha_t^* \exp(\tau_t/\alpha_t^*)$. $\square$

Proposition 3. The optimal policy that maximizes an expected additive-exponential utility can be obtained by solving the dynamic programming algorithm given by

$$\boldsymbol{\pi}_t(\boldsymbol{x}_t, w_t) = \begin{cases} \arg \max\limits_{\boldsymbol{u}_T \in U_T(\boldsymbol{x}_T)} \mathbb{E}_{\tilde{\boldsymbol{z}}_T}\left[-\delta_T \exp\left(-\dfrac{(1+\beta)w_T + g_T(\boldsymbol{x}_T, \boldsymbol{u}_T, \tilde{\boldsymbol{z}}_T)}{(1+\beta)^T \alpha_T}\right)\right] & t = T, \\ \arg \max\limits_{\boldsymbol{u}_t \in U_t(\boldsymbol{x}_t)} \mathbb{E}_{\tilde{\boldsymbol{z}}_t}\left[V_t^f\left(\boldsymbol{f}_t(\boldsymbol{x}_t, \boldsymbol{u}_t, \tilde{\boldsymbol{z}}_t), w_t, g_t(\boldsymbol{x}_t, \boldsymbol{u}_t, \tilde{\boldsymbol{z}}_t)\right)\right] & t \in \mathcal{T}^-, \end{cases}$$

$$\boldsymbol{\phi}_t(\boldsymbol{x}_{t+1}, w_t, r_t) = \arg \max_{b_t \in F_t(\boldsymbol{x}_{t+1}, w_t)} \left\{-\delta_t \exp\left(-\frac{r_t + b_t}{(1+\beta)^t \alpha_t}\right) + V_{t+1}^o\left(\boldsymbol{x}_{t+1}, (1+\beta)w_t - b_t\right)\right\},$$

where

$$V_t^o(\boldsymbol{x}_t, w_t) = \begin{cases} \max\limits_{\boldsymbol{u}_T \in U_T(\boldsymbol{x}_T)} \mathbb{E}_{\tilde{\boldsymbol{z}}_T}\left[-\delta_T \exp\left(-\dfrac{(1+\beta)w_T + g_T(\boldsymbol{x}_T, \boldsymbol{u}_T, \tilde{\boldsymbol{z}}_T)}{(1+\beta)^T \alpha_T}\right)\right] & t = T \\ \max\limits_{\boldsymbol{u}_t \in U_t(\boldsymbol{x}_t)} \mathbb{E}_{\tilde{\boldsymbol{z}}_t}\left[V_t^f\left(\boldsymbol{f}_t(\boldsymbol{x}_t, \boldsymbol{u}_t, \tilde{\boldsymbol{z}}_t), w_t, g_t(\boldsymbol{x}_t, \boldsymbol{u}_t, \tilde{\boldsymbol{z}}_t)\right)\right] & t \in \mathcal{T}^- \end{cases}$$

$$V_t^f(\boldsymbol{x}_{t+1}, w_t, r_t) = \max_{b_t \in F_t(\boldsymbol{x}_{t+1}, w_t)} \left\{ -\delta_t \exp\left(-\frac{r_t + b_t}{(1+\beta)^t \alpha_t} \right) + V_{t+1}^o\left(\boldsymbol{x}_{t+1}, (1+\beta)w_t - b_t\right) \right\}.$$

Proof. Following standard dynamic programming procedure, let $V_t^o(\boldsymbol{x}_t, w_t)$ be the maximal value of $\sum_{i=t}^T \mathbb{E}[-\delta_i \exp(-\tilde{c}_i/\alpha_i)]$ given $(\boldsymbol{x}_t, w_t)$ at the beginning of period t, $t \in \mathcal{T}$. Similarly, let $V_t^f(\boldsymbol{x}_{t+1}, w_t, r_t)$ be the maximal value of $\sum_{i=t}^T \mathbb{E}[-\delta_i \exp(-\tilde{c}_i/\alpha_i)]$ given w_t, r_t and $\boldsymbol{x}_{t+1}$ at the end of period t. We observe that

$$V_T^o(\boldsymbol{x}_t, w_t) \quad = \quad \max_{\boldsymbol{u}_T \in U_T(\boldsymbol{x}_T)} \mathbb{E}_{\tilde{\boldsymbol{z}}_T}\left[-\delta_T \exp\left(-\frac{(1+\beta)w_T + g_T(\boldsymbol{x}_T, \boldsymbol{u}_T, \tilde{\boldsymbol{z}}_T)}{(1+\beta)^T \alpha_T} \right) \right],$$

and we can obtain $V_t^f(\boldsymbol{x}_{t+1}, w_t, r_t)$ and $V_t^o(\boldsymbol{x}_t, w_t)$ by standard backward induction. $\square$

To obtain the optimal policy of Model (3.5), it remains to find the appropriate parameters $\boldsymbol{\delta}$ and $\boldsymbol{\alpha}$. We next present an algorithm for obtaining the optimal policy as follows.

Algorithm Main

Input: Initial multipliers $\boldsymbol{\lambda} \geq \boldsymbol{0}$ and parameters $\boldsymbol{\alpha} \geq \boldsymbol{0}$, $i = 1$.

repeat

Run **Algorithm Coordinate Descent** with input $(\boldsymbol{\lambda}, \boldsymbol{\alpha})$ and output $(\boldsymbol{h}, \boldsymbol{\alpha}, \boldsymbol{\Psi})$.

Update multipliers, $\boldsymbol{\lambda} := \max(\boldsymbol{\lambda} + d_i \boldsymbol{h}, 0)$

Update index $i := i + 1$.

until stopping criterion is met.

Output: $\boldsymbol{\Psi}$

Here $d_i > 0$ is a decreasing sequence of step sizes satisfying $\lim_{i\to\infty} d_i = 0$ and $\sum_{i=1}^{\infty} d_i = \infty$.

Algorithm Coordinate Descent

Input: Multipliers $\boldsymbol{\lambda} \geq \mathbf{0}$ and parameters $\boldsymbol{\alpha} \geq \mathbf{0}$.

repeat

Update $\boldsymbol{\Psi} := \arg\min_{\boldsymbol{\Psi}\in\mathcal{P}_H} L\left(\boldsymbol{\alpha}, \boldsymbol{\Psi}, \boldsymbol{\lambda}\right)$.

Update $\boldsymbol{\alpha} := \arg\min_{\boldsymbol{\alpha}\geq\mathbf{1}\epsilon} L\left(\boldsymbol{\alpha}, \boldsymbol{\Psi}, \boldsymbol{\lambda}\right)$.

until stopping criterion is met.

Output: $(\boldsymbol{h}, \boldsymbol{\alpha}, \boldsymbol{\Psi})$, where $h_t := \left(\alpha_t \mathbb{E}\left[\exp\left(-(\tilde{c}_t(\boldsymbol{\Psi}) - \tau_t)/\alpha_t\right)\right] - \alpha_t\right), t \in \mathcal{T}$.

Here the Lagrangian $L(\boldsymbol{\alpha}, \boldsymbol{\Psi}, \boldsymbol{\lambda})$ is defined as follows:

$$L(\boldsymbol{\alpha}, \boldsymbol{\Psi}, \boldsymbol{\lambda}) = \sum_{t\in\mathcal{T}} \left(\alpha_t + \lambda_t \left(\alpha_t \mathbb{E}\left[\exp\left(-(\tilde{c}_t(\boldsymbol{\Psi}) - \tau_t)/\alpha_t\right)\right] - \alpha_t\right)\right).$$

Observe that from Proposition 3, we can use dynamic programming to obtain the optimal policy $\boldsymbol{\Psi}$ that minimizes the Lagrangian for given $(\boldsymbol{\alpha}, \boldsymbol{\lambda})$. To obtain the optimal solution $\boldsymbol{\alpha}$ that minimizes the Lagrangian for given $(\boldsymbol{\lambda}, \boldsymbol{\Psi})$, we can do so by solving the univariate convex optimization problem,

$$\alpha_t = \arg\min \left\{ \alpha + \lambda_t \left(\alpha \mathbb{E}\left[\exp\left(-(\tilde{c}_t(\boldsymbol{\Psi}) - \tau_t)/\alpha\right)\right] - \alpha\right) \,\middle|\, \alpha \geq \epsilon \right\},$$

via bisection search techniques such as the Golden search methods (Kiefer

1953). To ensure that the algorithm converges to the optimal policy, we require a differentiability assumption on the Lagrangian, which is implied in our next assumption.

Assumption 2. Given $\boldsymbol{\alpha} > \mathbf{0}$ and $\boldsymbol{\lambda} \geq 0$, the Lagrangian $L(\boldsymbol{\alpha}, \boldsymbol{\Psi}, \boldsymbol{\lambda})$ is differentiable with respect to $\boldsymbol{\Psi}$ for all $\boldsymbol{\Psi} \in \mathcal{P}_H$.

Theorem 8. Under Assumptions 1 and 2, Algorithm Main returns an optimal policy that minimizes the CPRI criterion.

Proof. Algorithm Main is a standard subgradient optimization routine for obtaining the optimal multiplier solution, $\boldsymbol{\lambda}$ in the nondifferential dual function (see for instance, Bertsekas 1999, section 6.3). It calls upon Algorithm Coordinate Descent to obtain the subgradient $\boldsymbol{h}$. We first show for any input $\boldsymbol{\lambda}$, the limit point of the sequence $\{(\boldsymbol{\alpha}^{(i)}, \boldsymbol{\Psi}^{(i)})\}$, which is generated by Algorithm Coordinate Descent, minimizes $L(\boldsymbol{\alpha}, \boldsymbol{\Psi}, \boldsymbol{\lambda})$. We write $f(\boldsymbol{\alpha}, \boldsymbol{\Psi}) = L(\boldsymbol{\alpha}, \boldsymbol{\Psi}, \boldsymbol{\lambda})$ for ease of notation.

We first observe that $\boldsymbol{\alpha}^{(i)}$ is bounded below by $\mathbf{1}\epsilon$ and that $\boldsymbol{\Psi}^{(i)}$ is bounded, since we assume that the policy set is bounded. We will next show that the sequence $\{\boldsymbol{\alpha}^{(i)}\}$ is bounded above by $\hat{\boldsymbol{\alpha}}$, which is defined by

$$\hat{\alpha}_t = \begin{cases} \epsilon + 2 + 2\lambda_t \left(\sup\limits_{\boldsymbol{\Psi} \in \mathcal{P}_H} \mathbb{E}\left[\exp(-\tilde{c}_t(\boldsymbol{\Psi}) + \tau_t)\right] - 1 \right) & \text{if } \lambda_t \leq 0.5, \\ \epsilon + \max\left\{ 2 + 2\lambda_t \left(\sup\limits_{\boldsymbol{\Psi} \in \mathcal{P}_H} \mathbb{E}\left[\exp(-\tilde{c}_t(\boldsymbol{\Psi}) + \tau_t)\right] - 1 \right) \right. & \text{if } \lambda_t > 0.5, \\ \qquad \left. - \frac{\bar{c}_t}{\ln((\lambda_t - 0.5)/\lambda_t)} \right\} & \end{cases}$$

where $\bar{c}_t = \sup_{\boldsymbol{\Psi} \in \mathcal{P}_H} \inf\{v \mid \mathbb{P}(\tilde{c}_t(\boldsymbol{\Psi}) - \tau_t \leq v) = 1\}$ represents the maximal consumption premium in period t. Let us assume $\epsilon \in (0,1)$. For a given feasible policy $\boldsymbol{\Psi} \in \mathcal{P}_H$, suppose $\boldsymbol{\alpha} \geq \mathbf{1}\epsilon$ such that $\alpha_t > \hat{\alpha}_t$ for some $t \in \mathcal{T}$. We will show that $\boldsymbol{\alpha}$ is not the optimal solution that minimizes the Lagrangian for given $(\boldsymbol{\lambda}, \boldsymbol{\Psi})$. Indeed, we have:

$$\begin{aligned} &\alpha_t + \lambda_t \left(\alpha_t \exp\left(-(\tilde{c}_t(\boldsymbol{\Psi}) - \tau_t)/\alpha_t\right) - \alpha_t\right) \\ &\geq \alpha_t \left(1 + \lambda_t \exp\left(-\bar{c}_t/\alpha_t\right) - \lambda_t\right) = \upsilon_t. \end{aligned} \tag{3.9}$$

If $\lambda_t \leq 0.5$, then $\upsilon_t \geq 0.5\alpha_t$. If $\lambda_t > 0.5$ and $\bar{c}_t \leq 0$, then $\upsilon_t \geq \alpha_t(1+\lambda_t-\lambda_t) = \alpha_t$. Finally, if $\lambda_t > 0.5$ and $\bar{c}_t > 0$, then $-\bar{c}_t/\alpha_t \geq -\bar{c}_t/\hat{\alpha}_t \geq \ln\left((\lambda_t - 0.5)/\lambda_t\right)$ and $\upsilon_t \geq \alpha_t(1 + \lambda_t \times (\lambda_t - 0.5)/\lambda_t - \lambda_t) = 0.5\alpha_t$. We have:

$$\begin{aligned} &\alpha_t + \lambda_t \left(\alpha_t \exp\left(-(\tilde{c}_t(\boldsymbol{\Psi}) - \tau_t)/\alpha_t\right) - \alpha_t\right) \\ &\geq 0.5\alpha_t > 0.5\hat{\alpha}_t \geq 1 + \lambda_t \left(\sup_{\boldsymbol{\Psi}\in\mathcal{P}_H} \mathbb{E}\left[\exp(-\tilde{c}_t(\boldsymbol{\Psi}) + \tau_t)\right] - 1\right). \end{aligned}$$

Therefore, we can lower the value of the Lagrangian, $L(\boldsymbol{\alpha}, \boldsymbol{\Psi}, \boldsymbol{\lambda})$ by changing α_t to 1 and hence, $\boldsymbol{\alpha}^{(i+1)}$ must be bounded above by $\max\{\mathbf{1}, \hat{\boldsymbol{\alpha}}\}$. Since, $(\boldsymbol{\alpha}^{(i)}, \boldsymbol{\Psi}^{(i)})$ is a bounded sequence and there must exist at least one limit point.

Let $(\bar{\boldsymbol{\alpha}}, \bar{\boldsymbol{\Psi}})$ be a limit point of the sequence $\{(\boldsymbol{\alpha}^{(i)}, \boldsymbol{\Psi}^{(i)})\}$. Algorithm Coordinate Descent ensures that the following inequalities holds:

$$f(\boldsymbol{\alpha}^{(i)}, \boldsymbol{\Psi}^{(i)}) \geq f(\boldsymbol{\alpha}^{(i)}, \boldsymbol{\Psi}^{(i+1)}) \geq f(\boldsymbol{\alpha}^{(i+1)}, \boldsymbol{\Psi}^{(i+1)}), \tag{3.10}$$

for all i. Therefore, since $(\alpha^{(i)}, \boldsymbol{\Psi}^{(i)})$ is bounded, which implies $f(\boldsymbol{\alpha}^{(i)}, \boldsymbol{\Psi}^{(i)}) > -\infty$, the sequence $\{f(\boldsymbol{\alpha}^{(i)}, \boldsymbol{\Psi}^{(i)})\}$ is nonincreasing and converges to the limit point, $f(\bar{\boldsymbol{\alpha}}, \bar{\boldsymbol{\Psi}})$. It remains to prove that $(\bar{\boldsymbol{\alpha}}, \bar{\boldsymbol{\Psi}})$ minimizes f, which we will show by contradiction. Let $\{(\boldsymbol{\alpha}^{(i_j)}, \boldsymbol{\Psi}^{(i_j)}) \mid j = 0, 1, \cdots\}$ be a subsequence of $\{(\boldsymbol{\alpha}^{(i)}, \boldsymbol{\Psi}^{(i)})\}$ that converges to $(\bar{\boldsymbol{\alpha}}, \bar{\boldsymbol{\Psi}})$. Suppose there exists $\boldsymbol{\Psi}^o \in \mathcal{P}_H$ such that $\Delta = f(\bar{\boldsymbol{\alpha}}, \bar{\boldsymbol{\Psi}}) - f(\bar{\boldsymbol{\alpha}}, \boldsymbol{\Psi}^o) > 0$. Under Assumption 1, the function $f(\boldsymbol{\alpha}, \boldsymbol{\Psi})$ is differentiable with respect to $\boldsymbol{\alpha} > \mathbf{0}$ and $\boldsymbol{\Psi} \in \mathcal{P}_H$. Since continuity is implied by differentiability, we can find $\delta > 0$ such that $|f(\boldsymbol{\alpha}, \boldsymbol{\Psi}^o) - f(\bar{\boldsymbol{\alpha}}, \boldsymbol{\Psi}^o)| < \Delta/2$ for all $\boldsymbol{\alpha}$ with $\|\boldsymbol{\alpha} - \bar{\boldsymbol{\alpha}}\| \leq \delta$. Since $\{\boldsymbol{\alpha}^{(i_j)}\}$ converges to $\bar{\boldsymbol{\alpha}}$, we can find M_1 such that for all $j > M_1$, $||\boldsymbol{\alpha}^{(i_j)} - \bar{\boldsymbol{\alpha}}|| \leq \delta$ and therefore $|f(\boldsymbol{\alpha}^{(i_j)}, \boldsymbol{\Psi}^o) - f(\bar{\boldsymbol{\alpha}}, \boldsymbol{\Psi}^o)| < \Delta/2$. Moreover, since $\{f(\boldsymbol{\alpha}^{(i_j)}, \boldsymbol{\Psi}^{(i_j+1)})\}$ converges to $f(\bar{\boldsymbol{\alpha}}, \bar{\boldsymbol{\Psi}})$, there exists M_2 such that for all $j > M_2$, $|f(\boldsymbol{\alpha}^{(i_j)}, \boldsymbol{\Psi}^{(i_j+1)}) - f(\bar{\boldsymbol{\alpha}}, \bar{\boldsymbol{\Psi}})| < \Delta/2$. Therefore, for $j > \max\{M_1, M_2\}$, we have:

$$f(\bar{\boldsymbol{\alpha}}, \bar{\boldsymbol{\Psi}}) < f(\boldsymbol{\alpha}^{(i_j)}, \boldsymbol{\Psi}^{(i_j+1)}) + \Delta/2 \leq f(\boldsymbol{\alpha}^{(i_j)}, \boldsymbol{\Psi}^o) + \Delta/2 < f(\bar{\boldsymbol{\alpha}}, \boldsymbol{\Psi}^o) + \Delta,$$

which contradicts the assumption that $f(\bar{\boldsymbol{\alpha}}, \bar{\boldsymbol{\Psi}}) = f(\bar{\boldsymbol{\alpha}}, \boldsymbol{\Psi}^o) + \Delta$. Hence, we conclude that $f(\bar{\boldsymbol{\alpha}}, \bar{\boldsymbol{\Psi}}) \leq f(\bar{\boldsymbol{\alpha}}, \boldsymbol{\Psi})$, $\forall \boldsymbol{\Psi} \in \mathcal{P}_H$. Therefore, $\bar{\boldsymbol{\Psi}}$ minimizes $f(\bar{\boldsymbol{\alpha}}, \cdot)$. Under the assumption of differentiability and convexity, the optimality condition is equivalent to $\nabla_{\boldsymbol{\Psi}} f(\bar{\boldsymbol{\alpha}}, \bar{\boldsymbol{\Psi}})'(\boldsymbol{\Psi} - \bar{\boldsymbol{\Psi}}) \geq 0$ for all $\boldsymbol{\Psi} \in \mathcal{P}_H$ (see for instance, Bertsekas 1999, Proposition 2.1.2).

Similarly, suppose there exists $\boldsymbol{\alpha}^o \geq \mathbf{1}\epsilon$ such that $\Delta = f(\bar{\boldsymbol{\alpha}}, \bar{\boldsymbol{\Psi}}) - f(\boldsymbol{\alpha}^o, \bar{\boldsymbol{\Psi}}) > 0$. By continuity argument, we can find $\delta > 0$ such that $|f(\boldsymbol{\alpha}^o, \boldsymbol{\Psi}) - f(\boldsymbol{\alpha}^o, \bar{\boldsymbol{\Psi}})| < \Delta/2$ for all $\boldsymbol{\Psi}$ with $\|\boldsymbol{\Psi} - \bar{\boldsymbol{\Psi}}\| \leq \delta$. Since $\{\boldsymbol{\Psi}^{(i_j)}\}$ converges to $\bar{\boldsymbol{\Psi}}$, we can find M_1 such that for all $j > M_1$, $||\boldsymbol{\Psi}^{(i_j)} - \bar{\boldsymbol{\Psi}}|| \leq \delta$ and therefore $|f(\boldsymbol{\alpha}^o, \boldsymbol{\Psi}^{(i_j)}) - f(\boldsymbol{\alpha}^o, \bar{\boldsymbol{\Psi}})| < \Delta/2$. Moreover, since $\{f(\boldsymbol{\alpha}^{(i_j)}, \boldsymbol{\Psi}^{(i_j)})\}$

converges to $f(\bar{\boldsymbol{\alpha}}, \bar{\boldsymbol{\Psi}})$, there exists M_2 such that for all $j > M_2$, $|f(\boldsymbol{\alpha}^{(i_j)}, \boldsymbol{\Psi}^{(i_j)}) - f(\bar{\boldsymbol{\alpha}}, \bar{\boldsymbol{\Psi}})| < \Delta/2$. Therefore, for $j > \max\{M_1, M_2\}$, we have:

$$f(\bar{\boldsymbol{\alpha}}, \bar{\boldsymbol{\Psi}}) < f(\boldsymbol{\alpha}^{(i_j)}, \boldsymbol{\Psi}^{(i_j)}) + \Delta/2 \leq f(\boldsymbol{\alpha}^o, \boldsymbol{\Psi}^{(i_j)}) + \Delta/2 < f(\boldsymbol{\alpha}^o, \bar{\boldsymbol{\Psi}}) + \Delta,$$

which contradicts the assumption that $f(\bar{\boldsymbol{\alpha}}, \bar{\boldsymbol{\Psi}}) = f(\boldsymbol{\alpha}^o, \bar{\boldsymbol{\Psi}}) + \Delta$. Hence, we conclude that $f(\bar{\boldsymbol{\alpha}}, \bar{\boldsymbol{\Psi}}) \leq f(\boldsymbol{\alpha}, \bar{\boldsymbol{\Psi}})$, $\forall \boldsymbol{\alpha} \geq \mathbf{1}\epsilon$. Since $f(\boldsymbol{\alpha}, \boldsymbol{\Psi})$ is differentiable and convex in $\boldsymbol{\alpha}$, it implies by the optimality condition that $\nabla_{\boldsymbol{\alpha}} f(\bar{\boldsymbol{\alpha}}, \bar{\boldsymbol{\Psi}})'(\boldsymbol{\alpha} - \bar{\boldsymbol{\alpha}}) \geq 0$ for all $\boldsymbol{\alpha} \geq \mathbf{1}\epsilon$.

Combining the two above results, we have:

$$\nabla f(\bar{\boldsymbol{\alpha}}, \bar{\boldsymbol{\Psi}})' \left((\boldsymbol{\alpha}, \boldsymbol{\Psi}) - (\bar{\boldsymbol{\alpha}}, \bar{\boldsymbol{\Psi}})\right) \geq 0, \qquad \forall \boldsymbol{\alpha} \geq \mathbf{1}\epsilon, \boldsymbol{\Psi} \in \mathcal{P}_H.$$

Since $f(\boldsymbol{\alpha}, \boldsymbol{\Psi})$ is also jointly convex in $(\boldsymbol{\alpha}, \boldsymbol{\Psi})$, this implies that $(\bar{\boldsymbol{\alpha}}, \bar{\boldsymbol{\Psi}})$ minimizes $f(\cdot, \cdot)$.

Finally we show that the output of Algorithm Coordinate Descent, $\boldsymbol{h}$ is indeed a subgradient of $g(\cdot)$ at the input $\boldsymbol{\lambda}$. We observe $g(\cdot)$ is a pointwise minimum of a family of affine functions and hence is concave. Moreover, for any $\boldsymbol{\lambda}^o \geq \mathbf{0}$, we have:

$$\begin{aligned} g(\boldsymbol{\lambda}^o) - g(\boldsymbol{\lambda}) &= \min_{\boldsymbol{\alpha} \geq \mathbf{1}\epsilon, \boldsymbol{\Psi} \in \mathcal{P}_H} L(\boldsymbol{\alpha}, \boldsymbol{\Psi}, \boldsymbol{\lambda}^o) - L(\bar{\boldsymbol{\alpha}}, \bar{\boldsymbol{\Psi}}, \boldsymbol{\lambda}) \\ &\leq L(\bar{\boldsymbol{\alpha}}, \bar{\boldsymbol{\Psi}}, \boldsymbol{\lambda}^o) - L(\bar{\boldsymbol{\alpha}}, \bar{\boldsymbol{\Psi}}, \boldsymbol{\lambda}) \\ &= (\boldsymbol{\lambda}^o - \boldsymbol{\lambda})' \boldsymbol{h}. \end{aligned}$$

Therefore, $\boldsymbol{h}$ is a subgradient of $g(\cdot)$ at $\boldsymbol{\lambda}$. $\square$

3.3 Target-oriented inventory management

In this section, we study the joint inventory-pricing decision problem with financing control under the target-oriented CPRI decision criterion. The setup is similar to that in Chen et al. (2007), where a firm needs to determine the optimal replenishment and pricing policy spanning T periods. At the beginning of any period $t \in \mathcal{T}$, the inventory level x_t and the state of wealth w_t are observed. The decision maker then determines the selling price $p_t \in [\underline{p}_t, \bar{p}_t]$ and replenishes the inventory to the level $y_t \geq x_t$. The ordering cost, which consists of unit variable cost q_t and fixed ordering cost K_t, will be paid at the end of the period. The random demand is bounded and affected by the pricing decision p_t such that

$$\tilde{d}_t = D_t(p_t, \tilde{\boldsymbol{z}}_t) = \tilde{z}_t^1 - \tilde{z}_t^2 p_t,$$

where $\tilde{\boldsymbol{z}}_t = (\tilde{z}_t^1, \tilde{z}_t^2)$ are independently distributed nonnegative random variables satisfying $\mathbb{E}[\tilde{\boldsymbol{z}}_t] > \mathbf{0}$. The demand is realized near the end of the period and unsatisfied demand is backlogged. The inventory level is then updated as $x_{t+1} = y_t - d_t$ and the inventory cost $h_t(x_{t+1})$ is tabulated. The cost function, $h_t(x)$ is convex in x, represents holding cost if $x \geq 0$ and shortage cost otherwise. At the end of the planning horizon, $h_T(x)$ can be modified to include salvage values of unsold inventories.

Similar to Federgruen and Heching (1999), we assume

$$\lim_{x\to\infty}((q_t - q_{t+1}/(1+\beta))x + h_t(x)) = \lim_{x\to-\infty}(q_t x + h_t(x)) = \infty \qquad t \in \mathcal{T},$$

with $q_{T+1} = 0$ for simplicity. The uncertain income at the end of the period $t \in \mathcal{T}$, is hence given by

$$\tilde{r}_t = p_t D_t(p_t, \tilde{\boldsymbol{z}}_t) - h_t(y_t - D_t(p_t, \tilde{\boldsymbol{z}}_t)) - K_t\gamma(y_t - x_t) - q_t(y_t - x_t),$$

where

$$\gamma(x) = \begin{cases} 1 & \text{if } x > 0, \\ 0 & \text{otherwise.} \end{cases}$$

Similar to the general dynamic decision model discussed in Section 3.2, once the income r_t is realized, the firm determines the financing level $b_t \in F_t(x_{t+1}, w_t)$ in period $t \in \mathcal{T}^-$ to fulfill the desired level of consumption c_t and the state of wealth w_{t+1} is updated accordingly. At any point in time, we define the *asset value* as the total wealth including the value of inventories on hand. Specifically, at the beginning of period $t \in \mathcal{T}$, the asset value is $a_t = w_t + q_t x_t/(1+\beta)$ and near the end of the period just after the demand is resolved and before the financing decision if $t \in \mathcal{T}^-$, the asset value is $\bar{a}_t = (1+\beta)w_t + q_{t+1}x_{t+1}/(1+\beta)$. Note here we apply a discount factor to the inventory value because the unit variable cost q_t is taken at the end of the period.

Before presenting our results under the CPRI criterion, we first summarize in Table 3.2 the replenishment policies under various decision criteria based on risk sensitive additive utility functions. Under the umbrella of risk neutral models, in the absence of pricing decision, i.e., $\bar{p}_t = \underline{p}_t$, Scarf (1959) establishes the well-known optimality of (s, S) inventory policy in the presence of fixed ordering cost. The inventory replenishment strategy in period t is characterized by two parameters (s_t, S_t) such that if the inventory level x_t is below s_t, an order of size $S_t - x_t$ is made. Otherwise, no order is placed. A special case of this policy is the base-stock policy, in which $s_t = S_t$ is the base-stock level, which is optimal when $K_t = 0$. If pricing decision is available, Chen and Simchi-Levi (2004) show that (s, S, A, p) joint inventory-pricing policy is optimal. The inventory strategy in period t is characterized by two parameters (s_t, S_t) and a set $A_t \subseteq [s_t, (s_t + S_t)/2]$, which can be empty depending on the problem instance. Whenever $x_t < s_t$ or $x_t \in A_t$, an order of size $S_t - x_t$ is placed. Otherwise, no order is placed. The price depends on the initial inventory level at the beginning of the period. Taking into account of decision maker's risk aversion, Chen et al. (2007) show that under full financing, the optimal policy under an additive-exponential utility has the same structure as the classical risk neutral counterpart. Chen et al. (2007) also present the structural results for general additive concave utility function and under full financing and $K_t = 0$. They show that the optimal policy is one of base-stock in which the base-stock level depends solely on the asset value a_t at the beginning of period $t \in \mathcal{T}$.

We now present our results for the joint inventory-pricing control problem under the CPRI criterion.

Theorem 9. Under full financing, the optimal policies of the joint inventory-

Tab. 3.2: Summary of results under additive utility decision criteria.

	Price Not a Decision		*Price is a Decision*	
	$K_t = 0$	$K_t > 0$	$K_t = 0$	$K_t > 0$
Risk-neutral model	Base-stock	(s, S)	Base-stock list price	(s, S, A, p)
Exponential utility under full financing	Base-stock	(s, S)	Base-stock	(s, S, A, p)
Increasing & concave utility under full financing	Asset dependent base-stock	?	Asset dependent base-stock	?

pricing control problem under the CPRI criterion are as follows:

1. A base stock policy is optimal without fixed ordering cost.

2. An (s, S) policy is optimal in the absence of pricing control.

3. An (s, S, A, p) policy is optimal in the general case when pricing decision is allowed and the fixed ordering cost is positive.

Proof. We have established in Theorem 6 that the optimal policies of the inventory-pricing decision problems correspond to those that maximize an expected exponential utility of the total incomes. The structural results of these policies are derived by Chen et al. (2007). □

Unfortunately, structural results under constrained financing are limited and so we restrict ourselves to the case when where is no fixed ordering cost, which belongs to the class of convex dynamic decision problems. We consider a special type of financing restriction, which we call *asset constrained* *financing* where borrowing above the asset value is prohibited, that is,

$$F_t(x_{t+1}, w_t) = \{b \mid b \leq \underbrace{(1+\beta)w_t + q_{t+1}x_{t+1}/(1+\beta)}_{=\bar{a}_t}\} \qquad t \in \mathcal{T}^-. \tag{3.11}$$

Under this restriction, borrowing is limited by the asset value, which comprises the wealth and inventory value.

Theorem 10. Under asset constrained financing, asset dependent base-stock policies are optimal for inventory-pricing decision problems without fixed cost if the decision criterion is CPRI or an additive-exponential utility. In other words, the base-stock level, $S_t(a_t)$ depends solely on the asset value, $a_t = w_t + q_t x_t/(1+\beta)$ at the beginning of the period $t \in \mathcal{T}$.

Proof. Observe that the uncertain consumptions are

$$\tilde{c}_t = \begin{cases} (1+\beta)a_T + l_T(y_T, p_t, \tilde{\boldsymbol{z}}_T) & t = T \\ (1+\beta)a_t + l_t(y_t, p_t, \tilde{\boldsymbol{z}}_t) - a_{t+1} & t \in \mathcal{T}^-, \end{cases}$$

where the function $l_t : \Re^4 \mapsto \Re$ is defined by:

$$l_t(y, p, z_1, z_2) = (q_{t+1}/(1+\beta) - q_t)\, y + (p - q_{t+1}/(1+\beta))\,(z_1 - z_2 p) - h_t(y - z_1 + z_2 p) \qquad t \in \mathcal{T},$$

and it is obvious that $l_t(y, p, z_1, z_2)$ is jointly concave in y and p for any $z_2 \geq 0$.

Given the state (x_t, a_t) at the beginning of period $t \in \mathcal{T}$, let $V_t(x_t, a_t)$

be the maximal value of $\mathbb{E}[\sum_{i=t}^{T} -\delta_i \exp(-\tilde{c}_i/\alpha_i)]$. We have

$$V_T(x_T, a_T) = \max_{\substack{y_T \geq x_T \\ p_T \in [\underline{p}_T, \bar{p}_T]}} \mathbb{E}\left[-\delta_T \exp\left(-\frac{(1+\beta)a_T + l_T(y_T, p_T, \tilde{\boldsymbol{z}}_T)}{(1+\beta)^T \alpha_T}\right)\right].$$

Since $\tilde{z}_T^2$ is always nonnegative, $l_T(y, p, \tilde{\boldsymbol{z}}_T)$ is concave in (y, p) for all realizations. Therefore, the objective function in the above equation is concave in (y_T, p_T, a_T). Hence, as shown in Chen et al. (2007), Proposition 4, $V_T(x_T, a_T)$ is concave in (x_T, a_T), and is nondecreasing in a_T. Moreover, since

$$V_T(x_T, a_T) = \exp\left(-\frac{a_T}{(1+\beta)^{T-1}\alpha_T}\right) \times \max_{\substack{y_T \geq x_T \\ p_T \in [\underline{p}_T, \bar{p}_T]}} \mathbb{E}\left[-\delta_T \exp\left(-\frac{l_T(y_T, p_T, \tilde{\boldsymbol{z}}_T)}{(1+\beta)^T \alpha_T}\right)\right],$$

the assumption on h_T implies that there exists a finite constant $S_T \in \Re$ such that the optimal order-up-to level is $y_T^* = \max\{x_T, S_T\}$.

Assume for some $t+1 \in \mathcal{T}$, $V_{t+1}(x_{t+1}, a_{t+1})$ is concave in (x_{t+1}, a_{t+1}) and nondecreasing in a_{t+1}. Given all information before the decision b_t and under asset constrained financing, the financing constraint $b_t \in F_t(x_{t+1}, w_t)$ can be represented by $a_{t+1} \geq 0$. Therefore, we have

$$V_t(x_t, a_t) = \max_{\substack{y_t \geq x_t \\ p_t \in [\underline{p}_t, \bar{p}_t]}} \mathbb{E}\left[\max_{a_{t+1} \geq 0} \left\{\mu_t\left((1+\beta)a_t + l_t(y_t, p_t, \tilde{\boldsymbol{z}}_t) - a_{t+1}\right) + V_{t+1}\left(y_t - \tilde{z}_t^1 + \tilde{z}_t^2 p_t, a_{t+1}\right)\right\}\right],$$

where $\mu_t : \Re \mapsto \Re$ is defined by:

$$\mu_t(c) = -\delta_t \exp\left(-\frac{c}{(1+\beta)^t \alpha_t}\right).$$

Since μ_t is concave and nondecreasing, and $l_t(y_t, p_t, \boldsymbol{z})$ is concave in (y_t, p_t) for all $z_t^2 \geq 0$, we know that $\mu_t\left((1+\beta)a_t + l_t(y_t, p_t, \boldsymbol{z}_t) - a_{t+1}\right)$ is concave in (y_t, p_t, a_t, a_{t+1}) for all $z_t^2 \geq 0$ and is nondecreasing in a_t. Further, s-ince $V_{t+1}(\cdot, \cdot)$ is concave, it implies $V_{t+1}\left(y_t - z_t^1 + z_t^2 p_t, a_{t+1}\right)$ is concave in (y_t, p_t, a_{t+1}) and independent of a_t. Therefore, we conclude that

$$\mathbb{E}\left[\max_{a_{t+1}\geq 0}\left\{\mu_t\left((1+\beta)a_t + l_t(y_t, p_t, \tilde{\boldsymbol{z}}_t) - a_{t+1}\right) + V_{t+1}\left(y_t - \tilde{z}_t^1 + \tilde{z}_t^2 p_t, a_{t+1}\right)\right\}\right]$$

is concave in (y_t, p_t, a_t) and nondecreasing in a_t . Therefore, by the consump-tion on h_t, there exists finite $S_t \in \Re$, which depends on a_t but is independent from x_t, such that the optimal order-up-to level is $y_t^* = \max\{x_t, S_t\}$. More-over, from Proposition 4 in Chen et al. (2007), $V_t(x_t, a_t)$ is concave in (x_t, a_t), and it is obviously nondecreasing in a_t.

By induction, we know for all $t \in \mathcal{T}$, $V_t(x_t, a_t)$ is concave in (x_t, a_t) and nondecreasing in a_t, and there exists an optimal S_t, which is independent from x_t but may depends on a_t, such that the optimal order up to level is $y_t^* = \max\{x_t, S_t\}$. Hence, an asset dependent base-stock policy is optimal to the additive-exponential utility criterion.

The result under the CPRI criterion follows from Theorem 7. With $K = 0$, the consumptions can be verified to be convex in the general history dependent policy. Therefore, Theorem 7 shows that the optimal policy can be solved by maximizing an expected additive-exponential utility, and hence the optimal policies are asset dependent base-stock policy. $\square$

We summarize our results in Table 3.3.

Tab. 3.3: Summary of new contributions.

	Price Not a Decision		*Price is a Decision*	
	$K_t = 0$	$K_t > 0$	$K_t = 0$	$K_t > 0$
CPRI under full financing	Base stock	(s, S)	Base stock	(s, S, A, p)
CPRI under asset constrained financing	Asset dependent base-stock	?	Asset dependent base-stock	?
Exponential utility under asset constrained financing	Asset dependent base-stock	?	Asset dependent base-stock	?

3.4 Computational study

In this section, we present a numerical study on an inventory control problem without pricing decisions. We consider a stylized five period problem, $T = 5$ and there is no fixed ordering cost, $K_t = 0$. The inventory cost functions are

$$h_t(x) = h_t^+ \max\{x, 0\} + h_t^- \max\{-x, 0\} \qquad t \in \mathcal{T},$$

where h_t^+ represents the unit inventory holding cost and h_t^- is the unit shortage cost. The values of the input parameters are presented in Table 3.4. We assume discrete demands uniformly distributed in $[0, 100]$ and independent across periods. The system starts with $x_1 = w_1 = 0$, and we assume unrestricted financing and zero interest rate.

With the above setting, we compare the consumptions profiles generated by the CPRI model against the risk neutral and additive-exponential utility models. In our computational studies, we first obtain the optimal inventory policies under various decision criteria. The optimal policy for the additive-

Tab. 3.4: Input parameters of the inventory model.

unit ordering cost	$q_t = 3,\ t \in \mathcal{T}$
selling price	$p_t = 10,\ t \in \mathcal{T}$
unit holding cost	$h_t^+ = 1,\ t \in \mathcal{T}$
unit penalty cost	$h_t^- = 6,\ t \in \mathcal{T}$

exponential utility model is solved using the dynamic programming recursion described in Chen et al. (2007). Next, we use Monte Carlo simulations with 100, 000 independent trails to estimate the consumptions profiles. In Figure 3.1, we present the risky profile of the cash flows generated by the inventory system under the optimal risk neutral policy in the absence of financing. It is not surprising that the cash flows have large variability.

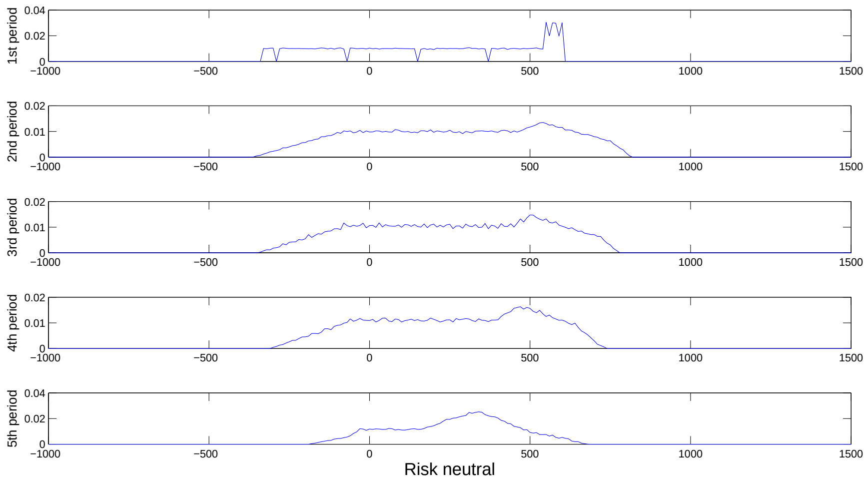

Fig. 3.1: Cash flows profile under optimal risk neutral policy.

3.4.1 CPRI versus Risk Neutral Model

In the first study, we compare the results between the CPRI model and the risk neutral model. For the CPRI model, we assume consumption targets in periods $t \in \mathcal{T}^-$ are zeros and we vary only the last period target, τ_T. Under an optimal FAT policy, consumptions will only appear in the last period. For the risk neutral model, whose objective function is indifferent to any financing decisions, we assume all operational cash flows in $t \in \mathcal{T}^-$ are saved and are used to finance consumption at the last period. Hence, in both experiments, it suffices to compare the consumptions at the final period.

To provide a fair comparison, we apply seven performance measures. The first two are expectation and standard deviation of the final consumptions. The third is the probability that the final consumptions will achieve the given target, and we call it *attainment probability* (AP). The fourth is expected loss (EL) relative to the target, and by normalizing it with the probability of loss we get the conditional expected loss (CEL) as the fifth measure. Finally, we consider the value at risk (VaR), i.e., the threshold loss that the loss with respect to the target does not exceed with a specific probability, at 95% and 99%. While the first three measures are intuitive, the last four measures are also widely adopted in financial risk management (Embrechts et al. 1997, Jorion 2006).

Table 3.5 shows the performance from the risk neutral model against the CPRI model. Observe that while yielding the consumptions profile with lower risk, the CPRI model only reduces the expectation with a relatively small fraction. In particular, with $\tau_T = 900$, the standard deviation is reduced by

Tab. 3.5: Performance of CPRI and risk neutral models.

τ_T	**Criterion**	**Performance measures**						
		Expected consumptions	AP	Standard deviation	EL	CEL	VaR @ 95%	VaR @ 99%
1200	Risk Neutral	1254.98	56.0%	419.46	143.54	326.54	653.05	943.64
	CPRI	1253.25	56.2%	406.18	139.09	317.33	634.80	921.61
1150	Risk Neutral	1254.98	60.5%	419.46	122.68	310.92	603.05	893.64
	CPRI	1247.71	60.8%	391.35	114.86	292.62	567.92	845.34
1100	Risk Neutral	1254.98	64.8%	419.46	104.02	295.55	553.05	843.64
	CPRI	1237.99	65.0%	375.04	93.82	268.40	502.90	772.24
1050	Risk Neutral	1254.98	68.9%	419.46	87.46	281.05	503.05	793.64
	CPRI	1222.13	69.1%	355.64	75.65	244.63	438.12	698.26
1000	Risk Neutral	1254.98	72.8%	419.46	72.89	268.07	453.05	743.64
	CPRI	1202.59	73.0%	336.54	60.37	223.29	377.22	627.94
950	Risk Neutral	1254.98	76.4%	419.46	60.21	255.59	403.05	693.64
	CPRI	1175.99	76.4%	315.18	47.68	202.41	319.00	559.31
900	Risk Neutral	1254.98	79.8%	419.46	49.27	243.71	353.05	643.64
	CPRI	1143.52	79.7%	292.41	37.20	183.00	264.77	491.63

AP=Attainment probability; EL=Expected Loss;
CEL=Conditional Expected Loss; VaR=Value at Risk.

30% and the mean loss, conditional mean loss, VaR @ 95%, and VaR @99% are all reduced by nearly 25%. And the only sacrifice is a 9% reduction in the expectation. The attainment probabilities from these two models are almost identical. Moreover, we can see from Table 3.5 that as the targets increase, the difference between the two models becomes less significant. The reason is that with high targets, the CPRI model is less risk averse and hence the result is similar to the risk neutral model.

3.4.2 CPRI versus Additive-Exponential Utility Model

We next compare the CPRI model against the additive-exponential utility model that minimizes the following criterion $\mathbb{E}\left[\sum_{t=1}^{5} \exp(-\tilde{c}_t/\alpha)\right]$ for some $\alpha > 0$. For the CPRI model, we apply the same consumption targets, $\tau_t = \tau$

for all $t \in \mathcal{T}$. In Figure 3.2, we present the consumptions profiles under the additive-exponential utility model as we vary $\alpha \in \{10, 100, 400\}$. In contrast, we show in Figure 3.3 the consumptions profiles under the CPRI model as we vary $\tau \in \{100, 150, 200\}$. We can see that while the additive exponential utility model yields uncertain consumptions in each period, the CPRI model yield deterministic consumption in the first four periods and relegate uncertainty in consumption to the last target.

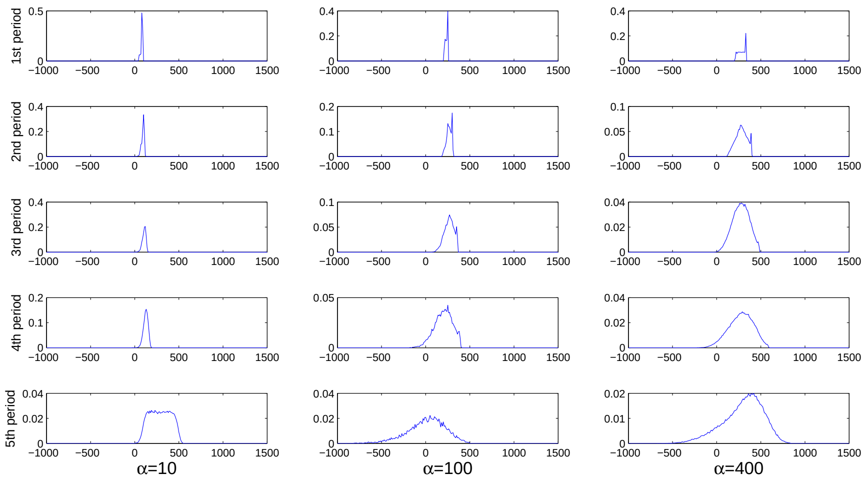

Fig. 3.2: Consumptions profiles under the additive-exponential utility model as α varies.

3.5 Conclusion

In this chapter, we propose a target-oriented decision model to help decision makers regulate their consumptions profile over time using some prescribed consumption targets. The model captures both the decision makers' risk aversion toward uncertain cash flow and their sensitive to the timing of the

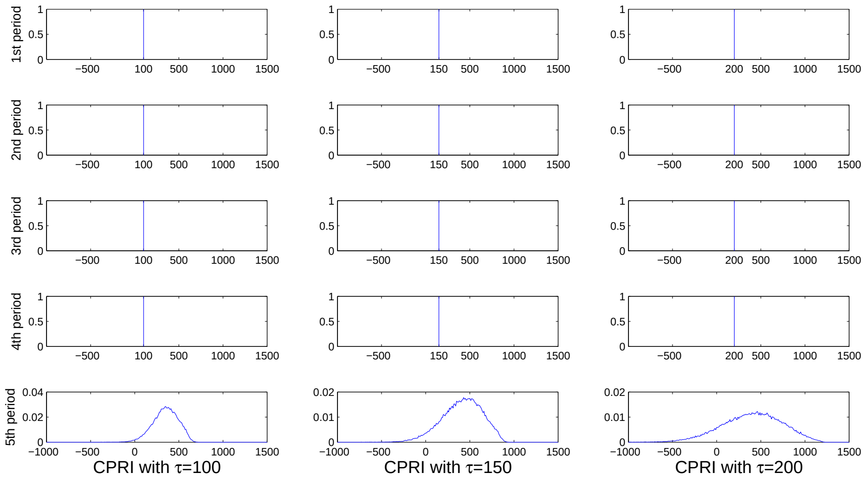

Fig. 3.3: Consumptions profiles under the CPRI model as τ varies.

resolution of uncertainties. By taking into account of the prescribed targets, the model highlights managers' practical concern in planning corporate activities: not only they need to meet everyday corporate consumptions such as sending out pay checks to employees, they are also typically concerned about whether they can achieve a target profit level at the end of the planning horizon.

With the CPRI criterion, we show that under full financing a FAT policy is optimal and accordingly, we can obtain the optimal operational policy by solving a modest number of dynamic programming problems. When financing is restricted, we show that for convex dynamic decision problems, the optimal policies correspond to those that maximize expected additive-exponential utilities. We also provide an algorithm to find the optimal policies.

Applying the CPRI criterion to the joint inventory-pricing decision problem, we identify the optimal inventory and pricing policies for the case with fixed ordering cost under full financing, and the policy structures for the case of zero fixed ordering cost under restricted financing. We also provide the policy structures when the objective function is additive-exponential utility and the fixed cost is zero. Finally, our numerical studies suggest that our target-oriented dynamic decisions model provides an interesting alternative for regulating consummations over time.

4. MANAGING UNDERPERFORMANCE RISK IN PROJECT PORTFOLIO SELECTION

In numerous organizations that own and manage projects, a problem of central importance is *selecting* which of the organization's available projects to accept and run. This decision is typically made either by a Project Management Office, or by a senior manager with experience as a project manager. Both statistical and anecdotal evidence suggest that doing the right projects is a big factor in doing projects right (Cooper et al. 2000). Indeed, well chosen projects are typically relatively easy to manage. Whereas, poorly selected projects are often dysfunctional and may compromise other projects by absorbing their resources.

Decisions to accept or reject projects can be made one project at a time, i.e. *sequentially* (Lochan 2010). In many organizations, available projects are initially screened according to various criteria such as their payback period and risk characteristics. Then, those projects that pass the initial screening are subjected to a more detailed evaluation that may include, for example, net present value and internal rate of return calculations. However, a *project portfolio approach* that simultaneously makes accept or reject decisions for all the available projects offers several important advantages over sequential selection (Kooragamage 2010). First, it more accurately models overall project portfolio issues, such as risk. Second, it more effectively utilizes the available resources. Third, it enables the modeling of correlation between uncertain project returns. Fourth, where there are interaction effects such as synergies between the projects, they can also be modeled effectively using the project portfolio approach. Finally, subject to the accuracy of available data, the project portfolio approach enables optimization of an overall objective for project selection.

A widely used approach for project selection is application of the deterministic zero one knapsack problem (Kellerer et al. 2004) and its extensions. An overview of project selection using such approaches is provided by Weber et al. (1990). Generalizations of the knapsack model (Fox et al. 1984, Dickinson et al. 2001) consider interaction effects that are limited to those arising from synergistic returns between pairs of projects. Keisler (2005) identifies synergies among larger groups of projects.

However, in many practical project selection problems, the applicability of deterministic optimization models is compromised by a high level of uncertainty in the return from each project at the time it is selected. A study of 1,015 projects by Pohl and Mihaljek (1992) documents this. Uncertainty in the return from a project may be classified into two main categories; uncertainty regarding its technical success, and uncertainty regarding its commercial success (MacMillan and McGrath 2002). The main sources of technical uncertainty include outcomes in research and development (e.g., for new products), in prototype testing (e.g., for safety testing of automobiles), and in regulatory approval (e.g., for new pharmaceutical products).

The main sources of commercial uncertainty include randomness in time to market (e.g., for seasonal products such as fashion items and toys), in the introduction of competitors' products (e.g., for consumer electronics), and in general economic factors (e.g., a recession). Hence, there is a need for a project portfolio methodology that achieves robust performance under uncertain project returns.

Keynes (1921), Knight (1921) and Ellsberg (1961) discuss the distinction between risk where probability distributions are known, and ambiguity where they are not. Because of the nonrecurring characteristics of projects (Newell et al. 2006), probability distributions for returns may not be known. For the project selection problem, classical robust optimization can be applied to maximize the worst case total return, when the returns of projects are described by an uncertainty set without distributional information (Soyster 1973, Ben-Tal and Nemirovski 1998). Kouvelis and Yu (1996) consider similar uncertainty sets, which can be represented by a convex hull of scenarios. However, neither approach considers the decision maker's ambiguity preference, and both are often viewed as overconservative. Moreover, the resulting models are typically highly intractable.

Bertsimas and Sim (2004) introduce an adjustable polyhedral uncertainty set where the decision maker can specify an ambiguity preference by defining a "budget of uncertainty". This approach avoids overconservatism and preserves the solvability of many underlying optimization models. However, it is sensitive to the budget of uncertainty parameter, which is apparently hard for many decision makers to estimate. Given the partially characterized distributions we consider, it is both possible and beneficial to avoid specifying this parameter.

In this chapter, we assume that exact probability distributions are not known. We consider the project selection problem under partially characterized distributions, or *distributional ambiguity*, about project returns. As in practice, these decisions are made subject to a budget and other constraints, such as portfolio diversification constraints and logical constraints between the projects (Beaujon et al. 2001, Loch et al. 2001). Our problem definition allows for correlation between the returns of different projects, which occurs naturally where projects use the same resources or are subject to the same external challenges. We also allow for interaction effects, for example synergies, between the returns of selected projects.

Our selection criterion is related to the concept of target based choice (Simon 1955), which argues that the main goal of most firms is not maximizing return but rather attaining a target return. Several descriptive studies of the risk behaviors of real world managers (Lanzillotti 1958, Mao 1970, Payne et al. 1980, 1981) confirm that aspiration levels drive decision making. In the same spirit, we allow the decision maker to stipulate a desired aspiration level or target return. Extending the riskiness index of Aumann and Serrano (2008), we propose the *underperformance riskiness index* for evaluating project portfolios. This index is the reciprocal of the absolute risk aversion (ARA) of an ambiguity averse individual with constant ARA who is indifferent between the target return with certainty and the uncertain portfolio return. Our model identifies the least risky project portfolio that meets the target. Our methodology for minimizing underperformance risk in project selection uses binary search on the ARA until indifference is achieved. At each value of the ARA, we apply a Benders decomposition method (Benders 1962) to solve the subproblem.

We compare our model with several classical approaches, both theoretically and computationally. We first identify several necessary properties for an appropriate approach to select projects. Based on those properties, we compare our model with traditional ones and identify its relative benefits. Our computational results show that our model finds more robust project portfolios than other approaches that are defined as benchmarks. Moreover, the model performs competitively with the classical approaches, even when measured by the various criteria that are optimized by those approaches. We also provide a simpler decision support tool for managers, in the form of a greedy heuristic, for use at the subproblem. This heuristic is shown computationally to identify project portfolios with near optimal underperformance risk. Finally, we discuss the implementability of our results for management decisions.

We briefly summarize our contributions to the literature, as follows.

1. We formally model interactions and correlations among project returns, which are distinct and important issues in the project selection problem. We introduce the concept of a "project bundle" to formulate the project interactions, and suggest a linear factor-based model to address the correlation issue.

2. We describe uncertainties using a two-stage process. The first stage uses Kullback-Leibler divergence to specify the probability of each possible scenario for the uncertain factor, and the second stage charac-

terizes each uncertain scenario using descriptive statistics such as the mean and bounds.

3. By adopting the underperformance riskiness index, we take into account the effect of target return, which is an important factor in real decision making but rarely considered in the literature.

4. Our computational studies show that the project selections found by minimizing the underperformance risk are more than competitive in achieving the target with those found by several classical benchmark approaches, and more robust in performance.

5. We provide a Benders decomposition algorithm to derive exact solutions, and a greedy heuristic with which managers can obtain a suboptimal solution. Our computational studies show that the greedy heuristic provides close to exact solutions.

Structure of the chapter. Section 4.1 defines the problem, describes the uncertainty and interaction structure, and presents our model. Section 4.2 contains a discussion of the solvability of various special cases of the problem. Section 4.3 describes an overall solution procedure. Section 4.4 describes a simple heuristic for solving the subproblem of the project selection model. In Section 4.5, we compare the performance of the solutions from Sections 4.3 and 4.4 against the benchmarks. Finally, Section 4.6 provides a conclusion, comments about implementability issues, and suggestions for future research.

4.1 Model Formulation

In Section 4.1.1, we provide our notation and define the project selection problem. Section 4.1.2 describes the modeling of interaction effects and uncertainty. Section 4.1.3 describes how we model distributional ambiguity. Section 4.1.4 describes our criterion for evaluating underperformance risk.

4.1.1 Notation and problem definition

Consider a set $\mathcal{N} = \{1, \dots, n\}$ of available projects. All projects are initially available. We use $\tilde{r}_j$ denotes the uncertain return of project j. Each project is either selected in full, or rejected. Let $y_j = 1$ if project j is selected, and $y_j = 0$ otherwise. We use boldface notation for matrices and vectors, e.g. $\boldsymbol{y} = (y_1, \dots, y_n)$.

We denote the total uncertain return of the selected projects by $\tilde{\pi}(\boldsymbol{y})$. The problem faced by the decision maker is

$$\max \quad \tilde{\pi}(\boldsymbol{y}) \tag{4.1}$$

$$\text{s.t.} \quad \boldsymbol{A}\boldsymbol{y} \leq \boldsymbol{b} \tag{4.2}$$

$$\boldsymbol{y} \in \{0, 1\}^n. \tag{4.3}$$

Objective (4.1) maximizes the total uncertain return (indeed we can not maximize this objective since there is uncertainty, a decision criterion is in need to map the random profit to a real value which can be maximized; we will discuss on the decision criterion later). Constraints (4.2) require that the selection of projects meets various deterministic restrictions discussed above, where $\mathbf{A}$ is a constraint matrix and $\mathbf{b}$ is a vector of resources. Constraint (4.3) ensures that each project is either accepted in full, or rejected. We let $\mathcal{Y} = \{\boldsymbol{y} : \boldsymbol{A}\boldsymbol{y} \leq \boldsymbol{b},\ \boldsymbol{y} \in \{0,1\}^n\}$. If there are no interactions between the returns of projects, then the total uncertain return of the selected projects is $\tilde{\pi}(\boldsymbol{y}) = \sum_{j=1}^{n} \tilde{r}_j y_j$. The next section generalizes this condition.

4.1.2 Interactions, uncertainty and correlation

Interactions among projects are common in practice. When several projects are implemented together, the total return can be greater or smaller than the sum of those projects' individual returns. For example, the implementation of two IT projects may create additional value through integration (Cho 2010). Yet, in the project selection literature, few papers consider this phenomenon. This is apparently due to the complexity which is introduced by modeling it. One approach (Fox et al. 1984) is to use the cross product of the binary decision variables. However, this formulation defines a nonlinear combinatorial optimization problem, and cannot clearly elicit the returns for interactions among more than two projects.

We consider interactions among the uncertain returns of a subset of projects with any cardinality. We define a *project bundle* $\beta \subseteq \mathcal{N}$ to consist of any number of projects that may have an interaction effect. By project screening, the manager can identify all project bundles. We define the *project bundle set*, $\mathcal{E}$, as follows.

Definition 9. The project bundle set, $\mathcal{E}$ is the minimal set with the following properties:

1. $\{i\} \in \mathcal{E}$, for $i \in \mathcal{N}$.

2. For $\beta_1, \beta_2 \in \mathcal{E}$, if $\beta_1 \cap \beta_2 \neq \emptyset$, then $\beta_1 \cup \beta_2 \in \mathcal{E}$.

The first property establishes consistency with the simple case without interaction effects. The second property shows that, for any two nondisjoint project bundles in $\mathcal{E}$, their union is a project bundle in $\mathcal{E}$. Hence, its return is well defined. This definition of project bundles generalizes earlier studies (Fox et al. 1984, Dickinson et al. 2001) by considering interactions among more than two projects.

We use an example to illustrate the project bundle set $\mathcal{E}$. Consider three available projects, $\mathcal{N} = \{1, 2, 3\}$. There is a synergistic effect when projects 1 and 2 are both selected, and when projects 2 and 3 are both selected. Therefore, $\{1, 2\}$ and $\{2, 3\}$ are project bundles in $\mathcal{E}$. According to Property 2 of Definition 9, $\{1, 2, 3\}$ is also an element in $\mathcal{E}$. Hence, the project bundle set $\mathcal{E} = \big\{\{1\}, \{2\}, \{3\}, \{1, 2\}, \{2, 3\}, \{1, 2, 3\}\big\}$.

The total uncertain return corresponding to selection $\boldsymbol{y}$ is given by

$$\tilde{\pi}(\boldsymbol{y}) = \sum_{\beta \in \mathcal{E}} \tilde{r}_\beta \Gamma(\beta, \boldsymbol{y}),$$

where $\tilde{r}_\beta$ represents the uncertain return from selecting β, and $\Gamma(\beta, \boldsymbol{y})$ is an indicator function for bundle β from the selection $\boldsymbol{y}$ defined by

$$\Gamma(\beta, \boldsymbol{y}) = \begin{cases} 1\ , & \text{if } y_j = 1, \text{ for } j \in \beta;\ \nexists \beta' \in \mathcal{E}, \beta' \supset \beta \text{ such that } y_j = 1, \text{ for } j \in \beta'; \\ 0\ , & \text{otherwise.} \end{cases}$$

Thus, if $\Gamma(\beta, \boldsymbol{y}) = 1$, project bundle β is a maximal subset of $\mathcal{N}$ under selection $\boldsymbol{y}$. In general, it is unrealistic for managers to identify interaction effects among all subsets of projects. Instead, they typically focus on a small number of significant interactions. Hence, $|\mathcal{E}|$ is not exponentially large in practice. Keisler (2005) describes the process used by project managers to estimate $\tilde{r}_\beta$.

Consistent with the above discussion of the sources of uncertainty, we assume that the uncertain return of each project bundle is an affine function of bounded, independently distributed *factors* $\tilde{z}_1, \ldots, \tilde{z}_K$, i.e.

$$\tilde{r}_\beta = r_\beta^0 + \sum_{k=1}^{K} r_\beta^k \tilde{z}_k,$$

where the factor coefficients $r_\beta^0, \ldots, r_\beta^K$ are known. Here we describe the correlation effect using a factor-based model instead of the typical covariance matrix. Our reasons are (a) estimating the covariance matrix from real data is difficult in general, especially for new or unique projects, (b) the linear factor-based model is motivated by practice since the performance of different project bundles depends on common factors in many cases, and (c) the linear-factor based model preserves the linear model structure and hence reduces the complexity of the solution process. In situations where historical data for similar projects is available, various statistical tools such as principal component analysis and linear regression can be used to calibrate the factors from the data. Even for a unique project, the factor-based model makes estimating the returns easier.

We denote by $\mathcal{V}$ the set of all attainable project returns, $\tilde{r} : \Omega \to \Re$, as follows

$$\mathcal{V} = \left\{ \tilde{v} \;\middle|\; \exists (v^0, \boldsymbol{v}) \in \Re^{K+1} \;:\; \tilde{v}(\omega) = v^0 + \sum_{k=1}^{K} v^k \tilde{z}_k(\omega), \;\; \omega \in \Omega \right\}.$$

The returns of two selected projects are correlated if they have one or more factors with nonzero factor coefficients in common. The total uncertain return from selection $\boldsymbol{y}$ is

$$\tilde{\pi}(\boldsymbol{y}) = \sum_{\beta \in \mathcal{E}} \tilde{r}_\beta \Gamma(\beta, \boldsymbol{y}) = \sum_{\beta \in \mathcal{E}} r_\beta^0 \Gamma(\beta, \boldsymbol{y}) + \sum_{k=1}^{K} \left(\sum_{\beta \in \mathcal{E}} r_\beta^k \Gamma(\beta, \boldsymbol{y}) \right) \tilde{z}_k.$$

4.1.3 Modeling risk and ambiguity

Since projects are typically unique (Project Management Institute 2004), there is a lack of historical information to elicit the actual distributions of their returns. Even with the availability of historical records, it is not always possible to determine exact distributions. The 2008 financial crisis provides many examples. Therefore, we do not assume knowledge of the exact probability distribution on $\mathcal{F}$. Instead, we permit ambiguity and assume that the true distribution, $\mathbb{P}$, lies in a family of distributions denoted by $\mathbb{F}$. The modeling of uncertainty in probability distributions is now well established in robust optimization; we refer interested readers to Ben-Tal et al. (2013) for recent developments. Our model for distributional ambiguity is motivated by *scenario building*, which is commonly used in practice for analyzing project payoffs (Hoffman 1985). Instead of specifying the exact probability and payoff associated with each scenario, which are often determined subjectively, we propose an ambiguity model over these parameters. For notational convenience, we temporarily suppress the subscript k in the uncertain factor $\tilde{z}_k$. Each uncertain factor $\tilde{z}$ can be decomposed into J mutually exclusive and collectively exhaustive scenarios, such that in the jth scenario, $\tilde{z}$ takes the value of $\tilde{\zeta}_j$ with probability p_j, where $\tilde{\zeta}_j$ is uncertain. For example, $\tilde{z}$ can take value $\tilde{\zeta}_1$ if the economic condition is good, with probability p_1; and it will take value $\tilde{\zeta}_2$ if the economic condition is weak. The set of possible vectors $\boldsymbol{p} = (p_1, \ldots, p_J)$ is denoted by $\mathcal{P} \subseteq \{\boldsymbol{p} \big| \ \boldsymbol{p}'\mathbf{1} = 1, \boldsymbol{p} \geq \mathbf{0}\}$.

Given the jth scenario, we assume that the probability distribution of $\tilde{\zeta}_j$ lies in some family of distributions $\mathbb{F}_j$. Further, we assume that $\mathbb{F}_j$ is an information set consisting of bounded support $\tilde{\zeta}_j \in \left[\underline{\zeta}_j, \bar{\zeta}_j\right]$, and mean support $\left[\underline{\mu}_j, \bar{\mu}_j\right] \subseteq \left[\underline{\zeta}_j, \bar{\zeta}_j\right]$, such that

$$\mathbb{F}_j = \left\{ \mathbb{P}_j \,\middle|\, \mathbb{P}_j \left(\tilde{\zeta}_j \in \left[\underline{\zeta}_j, \bar{\zeta}_j\right] \right) = 1, \mathbb{E}_{\mathbb{P}_j} \left(\tilde{\zeta}_j \right) \in \left[\underline{\mu}_j, \bar{\mu}_j\right] \right\}. \tag{4.4}$$

The family of distributions on $\tilde{z}$ is therefore characterized by $\mathcal{P}, \mathbb{F}_1, \ldots, \mathbb{F}_J$.

This two-stage approach for modeling distributional ambiguity generalizes the conventional scenario approach where the probabilities and payoffs are precisely given. This approach elicits the probability and other details of each uncertain scenario, and hence provides project managers with a decision support tool to analyze uncertain factors further and extract more accurate information. Next, we specify the objective function of the project selection problem for evaluating the preference of project portfolio returns, $\tilde{\pi}(\boldsymbol{y})$, under this model of uncertainty. We need to optimize over a possibly exponential number of feasible project portfolio returns, whose feasible space is not necessarily convex. Ideally, the objective function should be computable in polynomial time. Unfortunately, even in the absence of ambiguity, determining the distributions of $\tilde{\pi}(\boldsymbol{y})$, which involves the evaluation of a weighted sum of independently distributed random factors, is a computationally intractable problem (Khachiyan 1989, Nemirovski and Shapiro 2006). Moreover, the problem may be exacerbated by the presence of distributional ambiguity, where the evaluation of the objective function may require optimization over the family of distributions. Our computability requirement disqualifies many common criteria found in decision theory, including expected utility under general utility functions, and cumulative prospect theory (Tversky and Kahneman 1992). Similar concerns disqualify many coherent and convex risk measures (Artzner et al. 1999, Föllmer et al. 2004) from mathematical finance, including conditional value-at-risk (CVaR) of Rockafellar and Uryasev (2000) and the optimized certainty equivalent (OCE) of Ben-Tal and Teboulle (2007). Worst-case expected return is appealing due to its intuitiveness and tractability; however, while it considers ambiguity aversion, it ignores risk aversion. We refer interested readers to Knight (1921) and Hsu et al. (2005) for the distinction between risk and ambiguity. Mean-variance approaches, including Markowitz's (1959) and Roy's (1952) safety first, are computable but contradict the principle of monotonicity. That is, a portfolio with returns that outperform those in another portfolio in all scenarios may be ranked inferiorly. Hence, these approaches are also unsuitable.

The only utility function that we can tractably evaluate under our proposed model of uncertainty is the exponential utility function, which assumes that the decision maker has constant ARA and corresponds to a convex risk measure known as the entropic risk measure (Föllmer et al. 2004). However, a problem with this approach is how to elicit the ARA parameter from the decision maker, since it may be difficult to specify objectively. Moreover, behavioral paradoxes such as those of Allais (1953) and Ellsberg (1961), suggest the weakness of the expected utility paradigm in capturing human preference under uncertainty (Kahneman and Tversky 1979). In the next section, we present a closely related approach that resolves these issues.

4.1.4 Underperformance riskiness index

Aumann and Serrano (2008) propose an economic index of riskiness for evaluating a risky position, which is defined by the reciprocal of the ARA of an individual with constant ARA who is indifferent between taking and not taking the position. Incidentally, the riskiness index also falls within the framework of quasiconcave satisficing measures of Brown and Sim (2009) and acceptability indices of Cherny and Madan (2009). The index has useful attributes for risk management, including being positively homogenous and subadditive (Artzner et al. 1999, Föllmer et al. 2004). Moreover, Brown et al. (2012) show that this approach resolves important behavioral paradoxes.

Definition 10. Given a target, $\tau \in \Re$, the underperformance riskiness index (URI), $\rho_\tau : \mathcal{V} \to [0, \infty]$ is defined by

$$\rho_\tau(\tilde{v}) = \inf\left\{ \frac{1}{\alpha} \;\middle|\; C_\alpha(\tilde{v}) \geq \tau, \alpha > 0 \right\},$$

where the function $C_\alpha : \mathcal{V} \to \Re$ is defined by

$$C_\alpha(\tilde{v}) = \inf_{\mathbb{P}\in\mathbb{F}} \left(-\frac{1}{\alpha} \ln \left(\mathbb{E}_{\mathbb{P}} \left(\exp \left(-\alpha \tilde{v} \right) \right) \right) \right) = -\frac{1}{\alpha} \ln \left(\sup_{\mathbb{P}\in\mathbb{F}} \mathbb{E}_{\mathbb{P}} \left(\exp \left(-\alpha \tilde{v} \right) \right) \right).$$

We use the convention $\inf_{\emptyset}(\cdot) = \infty$.

The function $C_\alpha(\tilde{v})$ is the certainty equivalent of the uncertain return $\tilde{v}$ under the worst case exponential utility of Gilboa and Schmeidler (1989) with ARA parameter α. It has the following property.

Proposition 4. For any $\tilde{v} \in \mathcal{V}$, $C_\alpha(\tilde{v})$ is nonincreasing in $\alpha > 0$. Moreover,

$$\lim_{\alpha \downarrow 0} C_\alpha(\tilde{v}) = \inf_{\mathbb{P}\in\mathbb{F}} \mathbb{E}_{\mathbb{P}}(\tilde{v}). \tag{4.5}$$

Proof. For all $\alpha_1 > \alpha_2 > 0$,

$$\begin{aligned}
C_{\alpha_1}(\tilde{v}) &= \inf_{\mathbb{P}\in\mathbb{F}} -\frac{1}{\alpha_1} \ln \left(\mathbb{E}_{\mathbb{P}} \left(\exp(-\alpha_1 \tilde{v}) \right) \right) \\
&= \inf_{\mathbb{P}\in\mathbb{F}} -\frac{1}{\alpha_1} \ln \left(\mathbb{E}_{\mathbb{P}} \left(\left(\exp(-\alpha_2 \tilde{v}) \right)^{\frac{\alpha_1}{\alpha_2}} \right) \right) \\
&\leq \inf_{\mathbb{P}\in\mathbb{F}} -\frac{1}{\alpha_1} \ln \left(\left(\mathbb{E}_{\mathbb{P}} \left(\exp(-\alpha_2 \tilde{v}) \right) \right)^{\frac{\alpha_1}{\alpha_2}} \right) \\
&= \inf_{\mathbb{P}\in\mathbb{F}} -\frac{1}{\alpha_1} \frac{\alpha_1}{\alpha_2} \ln \left(\mathbb{E}_{\mathbb{P}} \left(\exp(-\alpha_2 \tilde{v}) \right) \right) \\
&= C_{\alpha_2}(\tilde{v}) \\
&\leq \inf_{\mathbb{P}\in\mathbb{F}} -\frac{1}{\alpha_2} \ln \left(\exp \left(\mathbb{E}_{\mathbb{P}}(-\alpha_2 \tilde{v}) \right) \right) \\
&= \inf_{\mathbb{P}\in\mathbb{F}} \mathbb{E}_{\mathbb{P}}(\tilde{v}),
\end{aligned}$$

where both inequalities follow from Jensen's inequality. Hence, to establish the equality (4.5), it suffices to show that $\lim_{\alpha \downarrow 0} C_\alpha(\tilde{v})$ is bounded below by $\inf_{\mathbb{P} \in \mathbb{F}} \mathbb{E}_{\mathbb{P}}(\tilde{v})$. Observe that $\tilde{v} \in \mathcal{V}$ is bounded, since it depends affinely on a finite set of bounded uncertain factors. Hence, there exists a $v > 0$ such that $|\tilde{v}(\omega)| \leq v$ for all $\omega \in \Omega$. Indeed, for $\mathbb{P} \in \mathbb{F}$ and $\alpha > 0$,

$$
\begin{aligned}
\mathbb{E}_{\mathbb{P}}\left(\exp(-\alpha\tilde{v})\right) &= 1 + \mathbb{E}_{\mathbb{P}}(-\tilde{v})\alpha + \sum_{n=2}^{\infty} \frac{1}{n!}\mathbb{E}_{\mathbb{P}}\left((-\tilde{v})^n\right)\alpha^n \\
&\leq 1 - \mathbb{E}_{\mathbb{P}}(\tilde{v})\alpha + \sum_{n=2}^{\infty} \frac{1}{n!}\mathbb{E}_{\mathbb{P}}\left(v^n\right)\alpha^n \\
&= 1 - \mathbb{E}_{\mathbb{P}}(\tilde{v})\alpha + \exp(v\alpha) - 1 - v\alpha \\
&= \exp(v\alpha) - \left(v + \mathbb{E}_{\mathbb{P}}(\tilde{v})\right)\alpha \\
&\leq \exp(v\alpha) - \left(v + \mu\right)\alpha,
\end{aligned}
$$

where $\mu = \inf_{\mathbb{P} \in \mathbb{F}} \mathbb{E}_{\mathbb{P}}(\tilde{v})$, and the first two equalities follow from Taylor's Theorem. Then

$$
\begin{aligned}
\lim_{\alpha \downarrow 0} C_\alpha(\tilde{v}) &= \lim_{\alpha \downarrow 0} \inf_{\mathbb{P} \in \mathbb{F}} -\frac{1}{\alpha} \ln\left(\mathbb{E}_{\mathbb{P}}\left(\exp(-\alpha\tilde{v})\right)\right) \\
&\geq \lim_{\alpha \downarrow 0} \left(-\frac{1}{\alpha} \ln\left(\exp(v\alpha) - \left(v + \mu\right)\alpha\right)\right) \\
&= \lim_{\alpha \downarrow 0} \left(-\frac{\exp(v\alpha)v - (v + \mu)}{\exp(v\alpha) - (v + \mu)\alpha}\right) \\
&= \mu,
\end{aligned}
$$

where the second equality follows from L'Hôpital's rule. □

The URI is the reciprocal of the highest ARA for which the target, τ, equals the certainty equivalent of $\tilde{v}$ under ambiguity aversion. The URI describes how risky $\tilde{v}$ is with respect to the target τ. In specifying the target τ, we assume that there exists a feasible project selection $\boldsymbol{y} \in \mathcal{Y}$ such that the expected return of the projects under ambiguity aversion exceeds the target, i.e. $\inf_{\mathbb{P}\in\mathbb{F}} \mathbb{E}_{\mathbb{P}} \left(\tilde{\pi}\left(\boldsymbol{y}\right)\right) \geq \tau$. If $\tilde{v}$ always achieves the target, i.e. $\tilde{v}(\omega) \geq \tau$ for all $\omega \in \Omega$, then $C_\alpha(\tilde{v}) \geq \tau$ for all $\alpha > 0$; hence, $\rho_\tau(\tilde{v}) = 0$, reflecting that the underperformance risk is zero. If $\tilde{v}$ never achieves the target, then $C_\alpha(\tilde{v}) < \tau$ for $\alpha > 0$; hence, $\rho_\tau(\tilde{v}) = \infty$.

In the case of known distributions, Brown et al. (2012) show that the URI model is consistent with second order stochastic dominance, and can also resolve several behavioral paradoxes that cannot be explained by expected utility preferences. In the case with distributional ambiguity, the worst case exponential utility model of Gilboa and Schmeidler (1989) can lead to conservative preferences that may strictly favor low probability rewards over ambiguity. Consider the following example:

- Project 1: Payoffs of $\$100,000$ w.p 0.01 and $\$0$ w.p 0.99,
- Project 2: Payoffs of either $\$100,000$ or $\$0$ with unknown probability.

Project 1 has a low probability of succeeding, while Project 2 has an unknown probability of succeeding. Due to the low success likelihood of Project 1, it is conceivable that Project 2 might be preferred in practice. However, under worst case utility preference, Project 1 is strictly preferred over Project 2. On the other hand, URI evaluates both projects as equally bad (URI $= \infty$), unless τ falls below $\$1,000$ in which case Project 1 is strictly preferred. In Brown et al. (2012), the URI model can be extended to encompass risk/ambiguity seeking behavior, so that Project 2 is strictly preferred if

τ exceeds \$1000. However, we do not consider this extension because we assume that investors are prudent and do not set targets that are unachievable in expectation.

We next present two characteristics of URI that are important in the context of risk management.

Proposition 5. For any risky positions, $\tilde{v}_1, \tilde{v}_2 \in \mathcal{V}$ and corresponding targets, $\tau_1, \tau_2 \in \Re$,

$$\rho_{\lambda\tau_1}(\lambda\tilde{v}_1) = \lambda\rho_{\tau_1}(\tilde{v}_1), \quad \text{for all } \lambda \geq 0;$$

$$\rho_{\tau_1+\tau_2}(\tilde{v}_1 + \tilde{v}_2) \leq \rho_{\tau_1}(\tilde{v}_1) + \rho_{\tau_2}(\tilde{v}_2).$$

Proof. According to the definition 10, for all $\lambda \geq 0$, we have

$$\begin{aligned}
\rho_{\lambda\tau_1}(\lambda\tilde{v}_1) &= \inf\left\{\frac{1}{\alpha}\Big|\ C_\alpha(\lambda\tilde{v}_1) \geq \lambda\tau_1, \alpha > 0\right\} \\
&= \inf\left\{\frac{1}{\alpha}\Big|\ -\frac{1}{\alpha}\ln\left(\sup_{\mathbb{P}\in\mathbb{F}}\mathbb{E}_{\mathbb{P}}\left(\exp\left(-\alpha\lambda\tilde{v}_1\right)\right)\right) \geq \lambda\tau_1, \alpha > 0\right\} \\
&= \inf\left\{\frac{1}{\alpha}\Big|\ -\frac{1}{\alpha\lambda}\ln\left(\sup_{\mathbb{P}\in\mathbb{F}}\mathbb{E}_{\mathbb{P}}\left(\exp\left(-\alpha\lambda\tilde{v}_1\right)\right)\right) \geq \tau_1, \alpha > 0\right\} \\
&= \lambda\inf\left\{\frac{1}{\alpha\lambda}\Big|\ -\frac{1}{\alpha\lambda}\ln\left(\sup_{\mathbb{P}\in\mathbb{F}}\mathbb{E}_{\mathbb{P}}\left(\exp\left(-\alpha\lambda\tilde{v}_1\right)\right)\right) \geq \tau_1, \alpha > 0\right\} \\
&= \lambda\rho_{\tau_1}(\tilde{v}_1).
\end{aligned}$$

Let $\rho_1 = \rho_{\tau_1}(\tilde{v}_1)$ and $\rho_2 = \rho_{\tau_2}(\tilde{v}_2)$. The result is trivially true if either ρ_1 or ρ_2 is infinite. Suppose not, then Proposition 4 implies that the following inequalities hold

$$\begin{array}{rcl}\sup_{\mathbb{P}\in\mathbb{F}} \mathbb{E}_{\mathbb{P}}\left(\exp\left(-\frac{\tilde{v}_1-\tau_1}{\rho_1+\epsilon_1}\right)\right) & \leq & 1, \quad \text{for all } \epsilon_1 > 0 \\ \sup_{\mathbb{P}\in\mathbb{F}} \mathbb{E}_{\mathbb{P}}\left(\exp\left(-\frac{\tilde{v}_2-\tau_2}{\rho_1+\epsilon_2}\right)\right) & \leq & 1, \quad \text{for all } \epsilon_2 > 0.\end{array}$$

Observe that for all $\epsilon_1, \epsilon_2 > 0$,

$$\begin{array}{rl} & \sup_{\mathbb{P}\in\mathbb{F}} \mathbb{E}_{\mathbb{P}}\left(\exp\left(-\frac{\tilde{v}_1+\tilde{v}_2-\tau_1-\tau_2}{\rho_1+\rho_2+\epsilon_1+\epsilon_2}\right)\right) \\ = & \sup_{\mathbb{P}\in\mathbb{F}} \mathbb{E}_{\mathbb{P}}\left(\exp\left(-\lambda\frac{\tilde{v}_1-\tau_1}{\rho_1+\epsilon_1} - (1-\lambda)\frac{\tilde{v}_2-\tau_2}{\rho_2+\epsilon_2}\right)\right) \\ \leq & \lambda \sup_{\mathbb{P}\in\mathbb{F}} \mathbb{E}_{\mathbb{P}}\left(\exp\left(-\frac{\tilde{v}_1-\tau_1}{\rho_1+\epsilon_1}\right)\right) + (1-\lambda)\sup_{\mathbb{P}\in\mathbb{F}} \mathbb{E}_{\mathbb{P}}\left(\exp\left(-\frac{\tilde{v}_2-\tau_2}{\rho_2+\epsilon_2}\right)\right) \\ \leq & 1, \end{array}$$

where $\lambda = \frac{\rho_1+\epsilon_1}{\rho_1+\rho_2+\epsilon_1+\epsilon_2} \in [0,1]$. The first inequality follows from convexity. Equivalently, we have

$$C_{1/(\rho_1+\rho_2+\epsilon)}(\tilde{v}_1+\tilde{v}_2) \geq \tau_1+\tau_2, \qquad \text{for all } \epsilon > 0.$$

Hence, $\rho_{\tau_1+\tau_2}(\tilde{v}_1+\tilde{v}_2) \leq \rho_1+\rho_2$. $\square$

Proposition 5 implies that when projects are merged, the collective underperformance risk in meeting the targets is no more than the sum of the underperformance risks of the individual projects in meeting their own targets. This result follows typical risk management practice in encouraging diversification.

From Definition 10, the project selection problem under the URI is

$$
\begin{aligned}
\rho_\tau = \min_{\alpha>0} \quad & 1/\alpha \\
\text{s.t.} \quad C_\alpha(\tilde{\pi}(\boldsymbol{y})) &\geq \tau \\
\boldsymbol{y} &\in \mathcal{Y}.
\end{aligned}
\tag{4.6}
$$

The first result simplifies $C_\alpha(\tilde{\pi}(\boldsymbol{y}))$.

Lemma 5. For all $\alpha > 0$, $C_\alpha(\tilde{\pi}(\boldsymbol{y}))$ can be decomposed into

$$
C_\alpha(\tilde{\pi}(\boldsymbol{y})) = \sum_{k=0}^{K} C_\alpha\left(\sum_{\beta\in\mathcal{E}} r_\beta^k \Gamma(\beta, \boldsymbol{y})\tilde{z}_k\right),
$$

where we let $\tilde{z}_0 = 1$ for simplicity.

Proof. Recall that $\tilde{z}_0, \ldots, \tilde{z}_K$ are independently distributed. Therefore,

$$
\begin{aligned}
C_\alpha(\tilde{\pi}(\boldsymbol{y})) &= -\frac{1}{\alpha}\ln\left(\sup_{\mathbb{P}\in\mathbb{F}} \mathbb{E}_\mathbb{P}\left(\exp\left(-\alpha\sum_{k=0}^{K}\left(\sum_{\beta\in\mathcal{E}} r_\beta^k\Gamma(\beta,\boldsymbol{y})\right)\tilde{z}_k\right)\right)\right) \\
&= -\frac{1}{\alpha}\ln\left(\prod_{k=0}^{K}\sup_{\mathbb{P}\in\mathbb{F}} \mathbb{E}_\mathbb{P}\left(\exp\left(-\alpha\sum_{\beta\in\mathcal{E}} r_\beta^k\Gamma(\beta,\boldsymbol{y})\tilde{z}_k\right)\right)\right) \\
&= \sum_{k=0}^{K} -\frac{1}{\alpha}\ln\left(\sup_{\mathbb{P}\in\mathbb{F}} \mathbb{E}_\mathbb{P}\left(\exp\left(-\alpha\sum_{\beta\in\mathcal{E}} r_\beta^k\Gamma(\beta,\boldsymbol{y})\tilde{z}_k\right)\right)\right) \\
&= \sum_{k=0}^{K} C_\alpha\left(\sum_{\beta\in\mathcal{E}} r_\beta^k\Gamma(\beta,\boldsymbol{y})\tilde{z}_k\right).
\end{aligned}
$$

□

To minimize the URI for the project selection problem, we perform binary search on the ARA parameter, α, to find the maximum value of α

that satisfies $C_\alpha(\tilde{v}) \geq \tau$, as justified by Proposition 4. We now consider the subproblem defined by a fixed value of $\alpha > 0$:

$$\begin{aligned} \max \quad & C_\alpha(\tilde{\pi}(\boldsymbol{y})) \\ \text{s.t.} \quad & \boldsymbol{y} \in \mathcal{Y}. \end{aligned} \tag{4.7}$$

4.2 Solvability

In this section, we investigate the computational solvability of several cases of the problem.

Remark 1. In some cases, the structure of the constraint matrix $\boldsymbol{Ay} \leq \boldsymbol{b}$ implies intractability, even for a deterministic version of the project selection problem. For example, under a simple knapsack constraint, the recognition version of the deterministic problem is binary *NP*-complete. Further, under arbitrary resource constraints, the recognition version of the deterministic problem is unary *NP*-complete.

We therefore focus on solvability results that derive from the uncertainty and interaction structure of the problem. Section 4.2.1 identifies a special case that is as solvable as the deterministic problem. Sections 4.2.2 and 4.2.3 use different assumptions about the uncertainty and interaction structure to establish negative results for two other special cases.

4.2.1 Independent returns without interactions

Proposition 6. If there are no interactions among the projects, and their returns are independent, then the objective function in problem (4.7) is a linear function of $\boldsymbol{y}$.

Proof. In the case of no interactions, the project bundle set is the same as the available project set, i.e., $\mathcal{E} = \mathcal{N}$, and for $i \in \mathcal{N}$, $\Gamma(i, \boldsymbol{y}) = y_i$.

For all $i \in \mathcal{N}$, let $\mathcal{I}_i = \{k |\ r_i^k \neq 0, k = 1, \ldots, K\}$, i.e., $\tilde{r}_i = r_i^0 + \sum_{k \in \mathcal{I}_i} r_i^k \tilde{z}_k$. Also, since the returns are independent, we have $\mathcal{I}_i \cap \mathcal{I}_j = \emptyset$, for $i, j \in \mathcal{N}$ and $i \neq j$. From Lemma 5,

$$\begin{aligned}
C_\alpha\left(\tilde{\pi}(\boldsymbol{y})\right) &= C_\alpha\left(\tilde{z}_0 \sum_{i \in \mathcal{N}} r_i^0 y_i\right) + \sum_{k=1}^{K} C_\alpha\left(\sum_{i \in \mathcal{N}} r_i^k y_i \tilde{z}_k\right) \\
&= \sum_{i \in \mathcal{N}} r_i^0 y_i + \sum_{i \in \mathcal{N}} \sum_{k \in \mathcal{I}_i} C_\alpha\left(r_i^k y_i \tilde{z}_k\right) \\
&= \sum_{i \in \mathcal{N}} \left(r_i^0 y_i + \sum_{k \in \mathcal{I}_i} C_\alpha\left(r_i^k y_i \tilde{z}_k\right)\right) \\
&= \sum_{i \in \mathcal{N}} \left(r_i^0 + \sum_{k \in \mathcal{I}_i} C_\alpha\left(r_i^k \tilde{z}_k\right)\right) y_i,
\end{aligned}$$

where the last equality holds since $y_i \in \{0, 1\}$. □

Proposition 6 shows that, without interactions and with independent uncertain returns, problem (4.7) retains the solvability of the problem $\max_{\boldsymbol{y} \in \mathcal{Y}} \boldsymbol{r}'\boldsymbol{y}$. For example, if the feasible set of $\boldsymbol{y}$ is a uniform matroid, i.e., $\mathcal{Y} = \{\boldsymbol{y} \in \{0,1\}^n |\ \sum_{k \in \mathcal{N}} y_k = m\}$, then problem (4.7) is optimally solved in $O(n \log n)$ time by sorting $\left(r_i^0 + \sum_{k \in \mathcal{I}_i} C_\alpha\left(\tilde{z}_k r_i^k\right)\right)$, $i = 1, \ldots, n$, in non-increasing order and selecting the first m projects. Alternatively, if the feasible set of $\boldsymbol{y}$ is defined by a simple knapsack constraint, i.e., $\mathcal{Y} = \{\boldsymbol{y} \in \{0,1\}^n \big| \sum_{k\in\mathcal{N}} a_k y_k = b\}$, then problem (4.7) is optimally solved in pseudopolynomial time (Kellerer et al. 2004).

Corollary 3. For the special case where project returns are independent and no interactions exist, problem (4.7) with a single budget constraint can be simplified to

$$\begin{aligned} \max \quad & \sum_{i\in\mathcal{N}} \left(r_i^0 + \sum_{k\in\mathcal{I}_i} C_\alpha \left(r_i^k \tilde{z}_k \right) \right) y_i \\ \text{s.t.} \quad & \boldsymbol{c}'\boldsymbol{y} \leq b, \\ & \boldsymbol{y} \in \{0,1\}^n. \end{aligned}$$

4.2.2 Correlated returns without interactions

We assume the absence of interactions, in order to focus solely on the role of correlation. Hence, $\mathcal{E} = \mathcal{N}$. We have the following result.

Proposition 7. If there are no interactions among the projects, then the recognition version of problem (4.7) with correlated returns under a uniform matroid is binary *NP*-complete.

Proof. By reduction from the following *NP*-Complete problem Gary and Johnson (1979).

Equal Cardinality Partition: Given a finite set $\mathcal{N}$ of even cardinality n, with size $c_k \in \mathcal{Z}^+$ for each $k \in \mathcal{N}$, determine if there exists a partition $\mathcal{N}_1, \mathcal{N}_2$ of $\mathcal{N}$ such that $|\mathcal{N}_1| = |\mathcal{N}_2| = n/2$ and $\sum_{k\in\mathcal{N}_1} c_k = \sum_{k\in\mathcal{N}_2} c_k$.

Under a uniform matroid, i.e., $\mathcal{Y} = \{\boldsymbol{y} \in \{0,1\}^n | \sum_{k\in\mathcal{N}} y_k = m\}$, we construct an instance of problem (4.7) with

$$\tau = \frac{1}{2}\sum_{k\in\mathcal{N}} c_k \tag{4.8}$$

$$\tilde{r}_k = \frac{2\tau}{n} + (c_k - \frac{2\tau}{n})\tilde{z}, \qquad k \in \mathcal{N} \tag{4.9}$$

$$m = \frac{n}{2}. \tag{4.10}$$

Equation (4.8) defines a specific target to achieve. Equation (4.9) defines a special type of uncertain return, where the return of each project is determined by a common uncertain factor $\tilde{z}$. We assume that $\tilde{z}$ is $+1$ or -1, with equal probability. Equation (4.10) implies that the only feasible solutions select $n/2$ projects. In this instance,

$$C_\alpha(\tilde{\pi}(\boldsymbol{y})) = -\frac{1}{\alpha}\ln\left(\frac{1}{2}\exp\left(-\alpha\sum_{k\in\mathcal{N}} c_k y_k\right) + \frac{1}{2}\exp\left(-\alpha\sum_{k\in\mathcal{N}} c_k(1-y_k)\right)\right).$$

We prove that there exists a selection for this instance such that the objective value is infinite, if and only if there exists a solution to Equal Cardinality Partition.

($\Rightarrow$) Suppose there exists a solution $\mathcal{N}_1$ and $\mathcal{N}_2$ to Equal Cardinality Partition such that $\sum_{k\in\mathcal{N}_1} c_k = \sum_{k\in\mathcal{N}_2} c_k = \tau$ and $|\mathcal{N}_1| = |\mathcal{N}_2| = n/2$. Then we set $y_i = 1, i \in \mathcal{N}_1$, and $y_i = 0, i \in \mathcal{N}_2$. Thus, for $\alpha > 0$,

$$C_\alpha(\tilde{\pi}(\boldsymbol{y})) = -\frac{1}{\alpha}\ln\left(\frac{1}{2}\exp\left(-\alpha\tau\right) + \frac{1}{2}\exp\left(-\alpha\tau\right)\right) = \tau.$$

Therefore, the objective value of this instance is infinite.

($\Leftarrow$) Suppose there exists a project selection $\boldsymbol{y}^*$ such that the objective value is infinite, i.e., $C_\alpha(\tilde{\pi}(\boldsymbol{y}^*)) \geq \tau$, for $\alpha > 0$. Since, the function $\exp(-\alpha x)$ is strictly convex in x for $\alpha > 0$, we have

$$\begin{aligned}
C_\alpha(\tilde{\pi}(\boldsymbol{y}^*)) &= -\frac{1}{\alpha}\ln\left(\frac{1}{2}\exp\left(-\alpha\sum_{k\in\mathcal{N}} c_k y_k^*\right) + \frac{1}{2}\exp\left(-\alpha\sum_{k\in\mathcal{N}} c_k(1-y_k^*)\right)\right) \\
&\leq -\frac{1}{\alpha}\ln\left(\exp\left(-\frac{1}{2}\alpha\left(\sum_{k\in\mathcal{N}} c_k y_k^* + \sum_{k\in\mathcal{N}} c_k(1-y_k^*)\right)\right)\right) \\
&= \frac{1}{2}\left(\sum_{k\in\mathcal{N}} c_k y_k^* + \sum_{k\in\mathcal{N}} c_k(1-y_k^*)\right) \\
&= \tau,
\end{aligned}$$

which implies that $C_\alpha(\tilde{\pi}(\boldsymbol{y}^*)) = \tau$. Furthermore, equality holds if and only if $\sum_{k\in\mathcal{N}} c_k y_k^* = \sum_{k\in\mathcal{N}} c_k(1-y_k^*) = \tau$. Hence, there exists a solution $\mathcal{N}_1 = \{k \mid y_k^* = 1, k \in \mathcal{N}\}$ and $\mathcal{N}_2 = \mathcal{N} \setminus \mathcal{N}_1$ to Equal Cardinality Partition. $\square$

4.2.3 Independent returns and interactions

Proposition 8. If there are interactions among the projects, then the recognition version of problem (4.7) with independent deterministic returns under a uniform matroid is unary *NP*-complete.

Proof. By reduction from the following unary *NP*-Complete problem (Gary and Johnson 1979).

Maximum Clique: Given an undirected graph $G = (V, E)$, with vertex set V, edge set E, and an integer m, does there exist a clique in G of total size at least m?

Under a uniform matroid, i.e., $\mathcal{Y} = \{\boldsymbol{y} \in \{0,1\}^n \big| \sum_{k \in \mathcal{N}} y_k = m\}$, we construct an instance of problem (4.7) where each project i corresponds to a vertex $i \in G$, and has deterministic return 1. For any pair of selected projects $(i,j) \in \mathcal{N}$, no interaction effect occurs if $(i,j) \in E$, and an interaction effect of -1 occurs if $(i,j) \notin E$. We show that this instance has a total return of at least m, if and only if the maximum clique problem has a solution.

($\Rightarrow$) Suppose there exists a maximum clique $G' \subset G$, where $|G'| \geq m$. Then, in the project selection problem, we select the projects that correspond to exactly m vertices of G'. By construction, the matroid constraint is satisfied, hence this selection is feasible. Moreover, the total deterministic return of the selected projects is m.

($\Leftarrow$) Suppose there exists a feasible selection $S \subset \mathcal{N}$ of projects with total return at least m. From feasibility, $|S| = m$. Since each project has deterministic return 1, and there are no positive interaction effects, the total return must be exactly m. Hence, the interaction effect among the selected projects must be 0. This implies that the set S corresponds to a solution to the maximum clique problem. $\square$

4.3 Algorithm

We now consider the subproblem (4.7) defined by a fixed value of $\alpha > 0$. The first result linearizes the indicator function, $\Gamma(\beta, \boldsymbol{y})$.

Lemma 6.

$$\sum_{\beta \in \mathcal{E}} r^k_\beta \Gamma(\beta, \boldsymbol{y}) = \boldsymbol{x}'\boldsymbol{v}^k, \quad k = 0, \ldots, K,$$

where $\boldsymbol{x}$ is determined from $\boldsymbol{y}$ by,

$$\begin{aligned} x_\beta &\leq y_i, && \beta \in \mathcal{E},\ i \in \beta \\ x_\beta + |\beta| &\geq \sum_{i \in \beta} y_i + 1, && \beta \in \mathcal{E} \\ \boldsymbol{x} &\in \{0,1\}^{|\mathcal{E}|}; \end{aligned}$$

$\boldsymbol{v}^k = (v^k_\beta)_{\beta \in \mathcal{E}}$, and $v^k_\beta = r^k_\beta - \sum_{\beta' \subset \beta,\ \beta' \in \mathcal{E}} v^k_{\beta'},\ k = 0, \ldots, K.$

Proof. Given $\boldsymbol{y}$, we note that for all $\beta \in \mathcal{E}$, $x_\beta = 1$ if and only if $y_i = 1$, for all $i \in \beta$. Let $\mathcal{G} = \{\beta \big| \ \Gamma(\beta, \boldsymbol{y}) = 1\}$, hence $\sum_{\beta \in \mathcal{E}} r^k_\beta \Gamma(\beta, \boldsymbol{y}) = \sum_{\beta \in \mathcal{G}} r^k_\beta$.

From the definition of $\boldsymbol{x}$, we observe that, for all $\beta \in \mathcal{E}$,

$$x_\beta = \begin{cases} 1, & \text{if there exists } \beta' \in \mathcal{G} \text{ such that } \beta \subseteq \beta'; \\ 0, & \text{otherwise.} \end{cases}$$

Therefore, for $k = 0, \ldots, K$,

$$\boldsymbol{x}'\boldsymbol{v}^k = \sum_{\beta \in \mathcal{E}} x_\beta v^k_\beta = \sum_{\beta' \in \mathcal{G}} \sum_{\beta \in \mathcal{E}, \beta \subseteq \beta'} v^k_\beta = \sum_{\beta' \in \mathcal{G}} (v^k_{\beta'} + \sum_{\beta \in \mathcal{E}, \beta \subset \beta'} v^k_\beta),$$

and from the definition of $v^k_{\beta'}$, we have

$$\boldsymbol{x}'\boldsymbol{v}^k = \sum_{\beta' \in \mathcal{G}} \left((r^k_{\beta'} - \sum_{\beta \in \mathcal{E}, \beta \subset \beta'} v^k_\beta) + \sum_{\beta \in \mathcal{E}, \beta \subset \beta'} v^k_\beta \right) = \sum_{\beta' \in \mathcal{G}} r^k_{\beta'} = \sum_{\beta \in \mathcal{E}} r^k_\beta \Gamma(\beta, \boldsymbol{y}).$$

□

Remark 2. The total return can be formulated as

$$\tilde{\pi}(\boldsymbol{y}) = \sum_{k=0}^{K} \boldsymbol{x}'\boldsymbol{v}^k \tilde{z}_k,$$

where $\boldsymbol{x}$ is determined from $\boldsymbol{y}$ according to Lemma 6.

Lemmas 5 and 6 imply that problem (4.7) can be reformulated as

$$\begin{array}{ll} \max & \sum_{k=0}^{K} C_\alpha\left(\boldsymbol{x}'\boldsymbol{v}^k \tilde{z}_k\right) \\ \text{s.t.} & \boldsymbol{x} \in \mathcal{X}, \end{array} \tag{4.11}$$

where $\mathcal{X}$ is defined as

$$\mathcal{X} = \left\{ \boldsymbol{x} \;\middle|\; \begin{array}{rcll} x_\beta & \leq & y_i, & \beta \in \mathcal{E},\ \ i \in \beta \\ x_\beta + |\beta| & \geq & \sum_{i\in\beta} y_i + 1, & \beta \in \mathcal{E} \\ \boldsymbol{y} & \in & \mathcal{Y} & \\ \boldsymbol{x} & \in & \{0,1\}^{|\mathcal{E}|} & \end{array} \right\}.$$

Next, we transform $C_\alpha(\boldsymbol{x}'\boldsymbol{v}^k \tilde{z}_k)$ into a piecewise linear function.

Proposition 9. For all $\boldsymbol{x} \in \mathcal{X}$,

$$C_\alpha(\boldsymbol{x}'\boldsymbol{v}^k \tilde{z}_k) = \min_{u \in \mathcal{U}(\boldsymbol{v}^k)} \left\{ C_\alpha(u\tilde{z}_k) + D_\alpha^k(u)(\boldsymbol{x}'\boldsymbol{v}^k - u) \right\}, \quad k = 0, \ldots, K,$$

where $D_\alpha^k(u)$ is a subgradient defined by

$$D_\alpha^k(u) = \frac{\mathbb{E}_{\mathbb{Q}}\left(\tilde{z}_k \exp(-\alpha u \tilde{z}_k)\right)}{\mathbb{E}_{\mathbb{Q}}\left(\exp(-\alpha u \tilde{z}_k)\right)},$$

$\mathbb{Q}$ is a probability distribution with $\mathbb{Q} \in \arg\sup_{\mathbb{P}\in\mathbb{F}} \mathbb{E}_{\mathbb{P}}\left(\exp(-\alpha u \tilde{z}_k)\right)$, and $\mathcal{U}(\boldsymbol{v}) = \{\boldsymbol{x}'\boldsymbol{v} \big| \ \boldsymbol{x} \in \mathcal{X}\}$.

Proof. For any $u \in \mathcal{U}(\boldsymbol{v}^k)$,

$$\begin{aligned}
C_\alpha(u\tilde{z}_k) - C_\alpha(\boldsymbol{x}'\boldsymbol{v}^k\tilde{z}_k) &= \frac{1}{\alpha}\ln\left(\sup_{\mathbb{P}\in\mathbb{F}} \mathbb{E}_{\mathbb{P}}\left(\exp\left(-\alpha\boldsymbol{x}'\boldsymbol{v}^k\tilde{z}_k\right)\right)\right) - \frac{1}{\alpha}\ln\left(\sup_{\mathbb{P}\in\mathbb{F}} \mathbb{E}_{\mathbb{P}}\left(\exp\left(-\alpha u\tilde{z}_k\right)\right)\right) \\
&\geq \frac{1}{\alpha}\left(\ln\left(\mathbb{E}_{\mathbb{Q}}\left(\exp\left(-\alpha\boldsymbol{x}'\boldsymbol{v}^k\tilde{z}_k\right)\right)\right) - \ln\left(\mathbb{E}_{\mathbb{Q}}\left(\exp\left(-\alpha u\tilde{z}_k\right)\right)\right)\right) \\
&\geq \frac{1}{\alpha}\left(\frac{\partial}{\partial\gamma}\ln\left(\mathbb{E}_{\mathbb{Q}}\left(\exp\left(-\alpha\gamma\tilde{z}_k\right)\right)\right)|_{\gamma=u} \times (\boldsymbol{x}'\boldsymbol{v}^k - u)\right) \\
&= -\frac{\mathbb{E}_{\mathbb{Q}}\left(\tilde{z}_k \exp\left(-\alpha u\tilde{z}_k\right)\right)}{\mathbb{E}_{\mathbb{Q}}\left(\exp\left(-\alpha u\tilde{z}_k\right)\right)}(\boldsymbol{x}'\boldsymbol{v}^k - u).
\end{aligned}$$

The first inequality follows from a property of the supremum. The second inequality follows from the convexity of the entropic function $\ln \mathbb{E}_{\mathbb{P}}\left(\exp(\cdot)\right)$. Therefore, for any $u \in \mathcal{U}(\boldsymbol{v}^k)$, we have

$$C_\alpha(\boldsymbol{x}'\boldsymbol{v}^k\tilde{z}_k) \leq C_\alpha(u\tilde{z}_k) + D_\alpha^k(u)(\boldsymbol{x}'\boldsymbol{v}^k - u). \tag{4.12}$$

Moreover, equality is achieved in (4.12) when $u = \boldsymbol{x}'\boldsymbol{v}^k$. □

Observe that the transformation of $C_\alpha(\boldsymbol{x}'\boldsymbol{v}^k\tilde{z}_k)$ into a piecewise linear function in Proposition 9 uses all the points $u \in \mathcal{U}(\boldsymbol{v}^k)$, and hence is exact. We evaluate $C_\alpha(u\tilde{z}_k)$ and $D_\alpha^k(u)$ in the statement of Proposition 9 by computing

$$\sup_{\mathbb{P}\in\mathbb{F}} \mathbb{E}_{\mathbb{P}}\left(\exp(-\alpha\tilde{z})\right) = \sup_{\boldsymbol{p}\in\mathcal{P}} \sum_{j=1}^{J} p_j \sup_{\mathbb{P}_j\in\mathbb{F}_j} \mathbb{E}_{\mathbb{P}_j}\left(\exp(-\alpha\tilde{\zeta}_j)\right) = \sup_{\boldsymbol{p}\in\mathcal{P}} \sum_{j=1}^{J} p_j\phi_j(\alpha),$$

where $\phi_j(\alpha) = \sup_{\mathbb{P}_j\in\mathbb{F}_j} \mathbb{E}_{\mathbb{P}_j}\left(\exp(-\alpha\tilde{\zeta}_j)\right)$, for all $\alpha > 0$. We first calculate $\phi_j(\alpha)$ under the family of distributions $\mathbb{F}_j$ in (4.4), and then compute $\sup_{\boldsymbol{p}\in\mathcal{P}} \sum_{j=1}^{J} p_j\phi_j(\alpha)$. We use an asymmetric measure of the difference between two probability distributions (Kullback and Leibler 1951). Given a reference distribution $\boldsymbol{q}$ and an uncertainty level θ, the real distribution $\boldsymbol{p}$ belongs to the set

$$\mathcal{P}_\theta = \left\{ \boldsymbol{p} \in \Re_+^J \,\middle|\, \sum_{j=1}^{J} p_j \ln\frac{p_j}{q_j} \le \theta, \sum_{j=1}^{J} p_j = 1 \right\}. \tag{4.13}$$

For example, if the uncertainty level is $\theta = 0$, we have $\boldsymbol{p} = \boldsymbol{q}$. Also, if $\theta \ge \max_j(-\ln q_j)$, $\boldsymbol{p}$ can be any distribution that satisfies the axioms of probability.

Proposition 10. Given the family of distributions (4.4) and the probability feasibility set (4.13), we have for $\alpha > 0$,

$$\phi_j(\alpha) = \sup_{\mathbb{P}_j\in\mathbb{F}_j} \mathbb{E}_{\mathbb{P}_j}\left(\exp\left(-\alpha\tilde{\zeta}_j\right)\right) = \frac{(\underline{\mu}_j - \underline{\zeta}_j)\exp(-\alpha\overline{\zeta}_j) + (\overline{\zeta}_j - \underline{\mu}_j)\exp(-\alpha\underline{\zeta}_j)}{\overline{\zeta}_j - \underline{\zeta}_j}$$

and $\mathbb{Q}$ is a probability distribution with $\mathbb{Q} \in \arg\sup_{\mathbb{P}\in\mathbb{F}} \mathbb{E}_{\mathbb{P}}\left(\exp(-\alpha\tilde{z})\right)$, where

$$\tilde{z} = \begin{cases} \underline{\zeta}_j, & \text{with probability } (\overline{\zeta}_j - \underline{\mu}_j)p_j^*/(\overline{\zeta}_j - \underline{\zeta}_j), \quad j = 1, \ldots, J \\ \overline{\zeta}_j, & \text{with probability } (\underline{\mu}_j - \underline{\zeta}_j)p_j^*/(\overline{\zeta}_j - \underline{\zeta}_j), \quad j = 1, \ldots, J, \end{cases}$$

$$p_j^* = \frac{q_j \exp\left(\phi_j(\alpha)/\lambda^*\right)}{\sum_{i=1}^J q_i \exp\left(\phi_i(\alpha)/\lambda^*\right)}, \qquad j = 1, \ldots, J,$$

and λ^* is the optimal solution found from $\inf_{\lambda>0}\left(\ \lambda \ln\left(\sum_{j=1}^J q_j \exp\left(\frac{\phi_j(\alpha)}{\lambda}\right)\right) + \theta\lambda\right)$ by using our binary search algorithm.

Proof. First, we calculate $\phi_j(\alpha)$. The information set of each possible scenario $\tilde{\zeta}_j$ has the same structure. Hence, we omit the subscript j, and calculate $\phi(\alpha) = \sup_{\mathbb{P}\in\mathbb{F}} \mathbb{E}_{\mathbb{P}}\left(\exp(-\alpha\tilde{\zeta})\right)$ by solving the linear program:

$$\begin{aligned} \phi(\alpha) = \max_f \quad & \mathbb{E}_f\left(\exp(-\alpha\zeta)\right) \\ \text{s.t.} \quad & \mathbb{E}_f(1) = 1 \\ & \mathbb{E}_f(\zeta) \leq \overline{\mu} \\ & \mathbb{E}_f(\zeta) \geq \underline{\mu} \\ & f(\zeta) \geq 0, \quad \zeta \in [\underline{\zeta}, \overline{\zeta}]. \end{aligned}$$

We consider f to consist of infinite dimensional decision variables indexed by

$[\underline{\zeta}, \overline{\zeta}]$. By linear programming duality, we have

$$\begin{aligned} \phi(\alpha) = \min \quad & y_0 + \overline{\mu} y_1 - \underline{\mu} y_2 \\ \text{s.t.} \quad & y_0 + \zeta y_1 - \zeta y_2 \;\geq\; \exp(-\alpha\zeta), \quad \forall \zeta \in [\underline{\zeta}, \overline{\zeta}] \\ & y_1, y_2 \;\geq\; 0, \end{aligned} \tag{4.14}$$

where the first constraint is equivalent to

$$\begin{aligned} y_0 \;&\geq \sup_{\zeta \in [\underline{\zeta}, \overline{\zeta}]} \{\exp(-\alpha\zeta) - (y_1 - y_2)\zeta\} \\ &= \max\left\{\exp(-\alpha\underline{\zeta}) - (y_1 - y_2)\underline{\zeta}\,,\ \exp(-\alpha\overline{\zeta}) - (y_1 - y_2)\overline{\zeta}\right\}. \end{aligned}$$

The equality follows from the convexity of $(\exp(-\alpha\zeta) - (y_1 - y_2)\zeta)$, which implies that its maximum value is achieved at an extreme point. Therefore, problem (4.14) can be written as

$$\phi(\alpha) = \min_{y_1, y_2 \geq 0} \max\left\{\exp(-\alpha\underline{\zeta}) + (\overline{\mu} - \underline{\zeta})y_1 + (\underline{\zeta} - \underline{\mu})y_2\,,\ \exp(-\alpha\overline{\zeta}) + (\overline{\mu} - \overline{\zeta})y_1 + (\overline{\zeta} - \underline{\mu})y_2\right\}. \tag{4.15}$$

Since the optimal values of y_1 and y_2 equate the two terms in (4.15), we have

$$y_1 - y_2 = \frac{\exp(-\alpha\underline{\zeta}) - \exp(-\alpha\overline{\zeta})}{\underline{\zeta} - \overline{\zeta}} < 0,$$

and

$$\phi(\alpha) = \min_{y_1 \geq 0} \left\{\exp(-\alpha\underline{\zeta}) + (\overline{\mu} - \underline{\mu})y_1 + (\underline{\zeta} - \underline{\mu})\frac{\exp(-\alpha\underline{\zeta}) - \exp(-\alpha\overline{\zeta})}{\overline{\zeta} - \underline{\zeta}}\right\}.$$

Similarly, $y_1 = 0$, and y_2 and y_0 can be derived. Hence, the optimal distribution is

$$\tilde{\zeta} = \begin{cases} \underline{\zeta}, & \text{with probability } (\overline{\zeta} - \underline{\mu})/(\overline{\zeta} - \underline{\zeta}), \\ \overline{\zeta}, & \text{with probability } (\underline{\mu} - \underline{\zeta})/(\overline{\zeta} - \underline{\zeta}). \end{cases}$$

We now calculate $\sup\limits_{\boldsymbol{p} \in \mathcal{P}_\theta} \sum_{j=1}^{J} p_j \phi_j(\alpha)$. For the real distribution $\boldsymbol{p} \in \mathcal{P}_\theta$, as defined by (4.13), the calculation of p_j^*, $j = 1, \ldots, J$, in the theorem statement follows from Nilim and El Ghaoui (September/October 2005). $\square$

Proposition 10 strengthens the inequality of Edmundson (1957) and Madansky (1959), since we only know the range of the mean, not its exact value.

From Proposition 10, we can calculate $C_\alpha(u\tilde{z}_k)$ and $D_\alpha^k(u)$, for $u \in \mathcal{U}(\boldsymbol{v}^k)$, in Proposition 9. Now, problem (4.11) can be reformulated as the mixed integer program

$$\begin{array}{lrcl} \max\limits_{\boldsymbol{x}, t_k} & \sum_{k=0}^{K} t_k & & \\ \text{s.t.} & t_k & \leq & C_\alpha(u\tilde{z}_k) + D_\alpha^k(u)(\boldsymbol{x}'\boldsymbol{v}^k - u), \quad k = 0, \ldots, K, \forall\, u \in \mathcal{U}(\boldsymbol{v}^k) \\ & \boldsymbol{x} & \in & \mathcal{X}. \end{array} \tag{4.16}$$

The size of $\mathcal{U}(\boldsymbol{v}^k)$ is large, and as discussed in Section 4.2, the existence of an efficient algorithm for searching it is unlikely. This justifies the following enumerative procedure.

Algorithm BD

1. For each $k = 0, \ldots, K$, choose a subset $\mathcal{U}^k = \{\boldsymbol{x}'\boldsymbol{v}^k\}$, for some $\boldsymbol{x} \in \mathcal{X}$.

2. Solve the following subproblem

$$\begin{array}{lrcl} \max\limits_{\boldsymbol{x},t_k} & \sum_{k=0}^{K} t_k & & \\ \text{s.t.} & t_k & \leq & C_\alpha(u\tilde{z}_k) + D_\alpha^k(u)(\boldsymbol{x}'\boldsymbol{v}^k - u), \quad k = 0,\ldots,K, \forall\, u \in \mathcal{U}^k \\ & \boldsymbol{x} & \in & \mathcal{X}, \end{array} \tag{4.17}$$

   and denote the solution by $\boldsymbol{x}^*$, and $t_k^*, k = 0,\ldots,K$.

3. If $t_k^* = C_\alpha(\boldsymbol{x}^{*\prime}\boldsymbol{v}^k)$, $k = 0,\ldots,K$, then output the optimal value and optimal solution $\boldsymbol{x}^*$, and stop.

4. For values of k such that $t_k^* > C_\alpha(\boldsymbol{x}^{*\prime}\boldsymbol{v}^k)$, add $(\boldsymbol{x}^{*\prime}\boldsymbol{v}^k)$ into $\mathcal{U}^k$, and go to Step 2.

Proposition 11. Algorithm BD finds an optimal solution to problem (4.16) in a finite number of steps.

Proof. When Algorithm BD terminates, it follows from Proposition 9 that

$$t_k^* = C_\alpha(\boldsymbol{x}^{*\prime}\boldsymbol{v}^k) \leq C_\alpha(u\tilde{z}_k) + D_\alpha^k(u)(\boldsymbol{x}^{*\prime}\boldsymbol{v}^k - u), \quad \forall\, u \in \mathcal{U}(\boldsymbol{v}^k).$$

Hence, $\boldsymbol{x}^*, t_k^*, k = 0,\ldots,K$ is feasible in problem (4.16). Since problem (4.17) is a relaxation of problem (4.16), $\boldsymbol{x}^*, t_k^*, k = 0,\ldots,K$ is also optimal in problem (4.16).

Note that $\mathcal{U}(\boldsymbol{v}^k)$ is a finite set for each k. Moreover, at each iteration, there exists at least one index k such that $\mathcal{U}^k$ increases. □

Remark 3. The need for Algorithm BD arises from the use of binary selection variables and the nonlinear objective function, not from our model choices of project bundles and the linear factor-based model.

4.4 Heuristic URI

In many situations, project selection is mainly constrained by a budget. For problems with this characteristic, we now describe a simple heuristic for the subproblem (4.7), that can be incorporated in the overall algorithm described in Section 4.3. This heuristic can easily be implemented on a spreadsheet to assist managerial decision making.

With a single budget constraint, the feasible project selection set is

$$\mathcal{Y} = \left\{ \boldsymbol{y} \;\middle|\; \begin{array}{rcl} \boldsymbol{c}'\boldsymbol{y} & \leq & b \\ \boldsymbol{y} & \in & \{0,1\}^n \end{array} \right\}.$$

Let $\boldsymbol{e}_i$ represent an n dimensional vector, where the ith element is one, and the others are zero. We describe a simple heuristic for problem (4.7).

Heuristic URI

Input: ARA parameter α.

Output: Heuristic solution $\boldsymbol{y}_\alpha^G$, and its certainty equivalent C_α^G.

1. Start with an empty selection, $\boldsymbol{y} = \boldsymbol{0}$;

   Set $\bar{c} = 0$, to represent the total cost of the currently selected projects.

2. Let $\mathcal{I} = \{i \,|\, i \in \mathcal{N},\ y_i = 0,\ \bar{c} + c_i \leq b\}$. If $\mathcal{I} = \emptyset$, then go to Step 5.

3. Find

$$j \in \arg\max \left\{ \frac{C_\alpha\left(\tilde{\pi}\left(\boldsymbol{y}+\boldsymbol{e}_i\right)\right) - C_\alpha\left(\tilde{\pi}(\boldsymbol{y})\right)}{c_i}, \qquad i \in \mathcal{I} \right\}.$$

4. If $C_\alpha\left(\tilde{\pi}(\boldsymbol{y}+\boldsymbol{e}_j)\right) > C_\alpha\left(\tilde{\pi}(\boldsymbol{y})\right)$, then select the jth project, set $\boldsymbol{y} = \boldsymbol{y}+\boldsymbol{e}_j$ and $\overline{c} = \overline{c} + c_j$, and go to Step 2; otherwise, go to Step 5.

5. Output $\boldsymbol{y}_\alpha^G = \boldsymbol{y}$ and $C_\alpha^G = C_\alpha\left(\tilde{\pi}(\boldsymbol{y})\right)$, and stop.

Remark 4. Since $\forall\, i \in \mathcal{N}$ with $y_i = 0$,

$$r_i^0 + \sum_{k \in \mathcal{I}_i} C_\alpha\left(r_i^k \tilde{z}_k\right) = C_\alpha\left(\tilde{\pi}(\boldsymbol{y}+\boldsymbol{e}_i)\right) - C_\alpha\left(\tilde{\pi}(\boldsymbol{y})\right),$$

Heuristic URI is the standard greedy rule for the deterministic knapsack problem (Kellerer et al. 2004).

We use a simple example, which considers both interactions and correlation, to illustrate the steps of Heuristic URI. Six projects are available with equal cost, to develop three products A, B and C. The project data appears in Table 4.1. Because of the limited budget, no more than three projects can be selected. For each product, there are two available projects, but selecting both of them results in diseconomies of scale. Thus, the total return is 20% less than the sum of the two individual returns. Furthermore, the uncertain return of each project depends affinely on two random factors. The first factor, for example technical issues in the project, influences only the project itself. The second factor, for example the state of the economy, potentially influences all projects. We define the decision variables $\boldsymbol{y} \in \{0,1\}^6$, and project bundle set $\mathcal{E} = \{\{1\},\{2\},\{1,2\},\{3\},\{4\},\{3,4\},\{5\},\{6\},\{5,6\}\}$. Table 4.1 shows the returns for each project bundle.

Product	Project Bundle	Return
A	1	$\tilde{r}_{\{1\}} = \tilde{z}_1 + \tilde{z}_7$
	2	$\tilde{r}_{\{2\}} = \tilde{z}_2 + \tilde{z}_7$
	{1,2}	$\tilde{r}_{\{1,2\}} = 0.8(\tilde{r}_{\{1\}} + \tilde{r}_{\{2\}})$
B	3	$\tilde{r}_{\{3\}} = \tilde{z}_3 + \tilde{z}_7$
	4	$\tilde{r}_{\{4\}} = \tilde{z}_4 + \tilde{z}_7$
	{3,4}	$\tilde{r}_{\{3,4\}} = 0.8(\tilde{r}_{\{3\}} + \tilde{r}_{\{4\}})$
C	5	$\tilde{r}_{\{5\}} = \tilde{z}_5 + \tilde{z}_7$
	6	$\tilde{r}_{\{6\}} = \tilde{z}_6 + \tilde{z}_7$
	{5,6}	$\tilde{r}_{\{5,6\}} = 0.8(\tilde{r}_{\{5\}} + \tilde{r}_{\{6\}})$

Tab. 4.1: Project Bundle Data in Heuristic URI Example.

Each uncertain factor is independent of the others, and the value of $\tilde{z}_i, i = 1, \ldots, 7$, is either $\underline{z}_i$ or $\overline{z}_i$, as shown in Table 4.2, with equal probability.

i	1	2	3	4	5	6	7
$\underline{z}_i$	0	-10	-20	-30	-40	-50	0
$\overline{z}_i$	20	40	60	80	120	140	10

Tab. 4.2: Factor Returns in Heuristic URI Example.

We set $\alpha = 0.0073$ as a trial value. Table 4.3 shows the calculations of Heuristic URI for the example.

For any given $\alpha > 0$, we use Heuristic URI to solve the subproblem, and then apply binary search to find the optimal value of α in (4.6). Thus, if $\tau \leq C_\alpha^G = C_\alpha(\tilde{\pi}(\boldsymbol{y})) = 59.59$, the trial value of α is increased; otherwise, it is decreased.

Iteration	Heuristic Steps	Calculations						j
1	Initialization	$\boldsymbol{y} = (0,0,0,0,0,0)$,		$\bar{c} = 0$,		$C_\alpha(\tilde{\pi}(\boldsymbol{y}))$=0		5
	Feasible Projects	$\mathcal{I} = \{1,2,3,4,5,6\}$						
	i	1	2	3	4	5	6	
	$C_\alpha(\tilde{\pi}(\boldsymbol{y}+\boldsymbol{e}_i))$	14.54	17.61	19.07	19.01	**22.49**	18.95	
2	Update	$\boldsymbol{y} = (0,0,0,0,1,0)$,		$\bar{c} = 1$,		$C_\alpha(\tilde{\pi}(\boldsymbol{y})) = 22.49$		3
	Feasible Projects	$\mathcal{I} = \{1,2,3,4,6\}$						
	i	1	2	3	4	6		
	$C_\alpha(\tilde{\pi}(\boldsymbol{y}+\boldsymbol{e}_i))$	36.85	39.92	**41.38**	41.32	40.81		
3	Update	$\boldsymbol{y} = (0,0,1,0,1,0)$,		$\bar{c} = 2$,		$C_\alpha(\tilde{\pi}(\boldsymbol{y})) = 41.38$		6
	Feasible Projects	$\mathcal{I} = \{1,2,4,6\}$						
	i	1	2	4	6			
	$C_\alpha(\tilde{\pi}(\boldsymbol{y}+\boldsymbol{e}_i))$	55.55	58.62	55.17	**59.59**			
4	Update	$\boldsymbol{y} = (0,0,1,0,1,1)$,		$\bar{c} = 3$,		$C_\alpha(\tilde{\pi}(\boldsymbol{y})) = 59.59$		
	Feasible Projects	$\mathcal{I} = \emptyset$, stop.						
	Output	$\boldsymbol{y}_\alpha^G = \boldsymbol{y} = (0,0,1,0,1,1)$, $C_\alpha^G = C_\alpha(\tilde{\pi}(\boldsymbol{y})) = 59.59$						

Tab. 4.3: Example Calculations using Heuristic URI.

4.5 Computational Studies

In Section 4.5.1, we describe several benchmark approaches that can be used for project selection, and how to solve them. Section 4.5.2 contains a comparative study of the URI and Heuristic URI against the benchmark approaches. Section 4.5.3 provides a sensitivity analysis of the performance of the URI. Finally, Section 4.5.4 studies the robustness of the URI.

4.5.1 Benchmark selection approaches

In this set of experiments, we assume for both the URI approaches and the benchmarks that the probability distributions of the factors are known. In Section 4.5.4, we show that this assumption does not significantly affect URI performance. We now describe several benchmark selection approaches for comparison with the URI.

Expected return

We consider maximizing expected return as the objective. Based on the formulation of total return in Remark 2, we need to solve the following problem.

$$\max_{\boldsymbol{x}\in\mathcal{X}} \mathbb{E}\left[\sum_{k=0}^{K}\boldsymbol{x}'\boldsymbol{v}^k\tilde{z}_k\right],$$

which is equivalent to

$$\max_{\boldsymbol{x}\in\mathcal{X}} \sum_{k=0}^{K}\boldsymbol{x}'\boldsymbol{v}^k\mathbb{E}\left(\tilde{z}_k\right). \tag{4.18}$$

Problem (4.18) is a deterministic knapsack problem, which is computationally easy to solve for large problems.

Underperformance probability

We consider the objective of minimizing the probability that the return will fall below a given target, τ. This model is

$$\min_{\boldsymbol{x}\in\mathcal{X}} \mathbb{P}\left(\sum_{k=0}^{K}\boldsymbol{x}'\boldsymbol{v}^k\tilde{z}_k < \tau\right).$$

However, optimization of this model is a highly intractable problem. We solve it using simulation. We generate a sample consisting of M independent random factor scenarios. Each scenario i contains a realization of independently generated random factors $z_1^i, \ldots, z_K^i$. Given $\boldsymbol{v}^0, \ldots, \boldsymbol{v}^K$, the problem of minimizing the underperformance probability is

$$
\begin{aligned}
\min \quad & \frac{1}{M}\sum_{i=1}^{M} I_i \\
\text{s.t.} \quad & I_i \ \geq \ \frac{\tau - \sum_{k=0}^{K} \boldsymbol{x}'\boldsymbol{v}^k z_k^i}{G}, \quad i = 1,\dots,M \\
& I_i \ \in \ \{0,1\}, \qquad\qquad\qquad i = 1,\dots,M \\
& \boldsymbol{x} \ \in \ \mathcal{X},
\end{aligned}
\tag{4.19}
$$

where G is a scalar which is sufficiently large that $\dfrac{\tau - \sum_{k=0}^{K} \boldsymbol{x}'\boldsymbol{v}^k z_k^i}{G} \in (-1,1)$. If the sample size M is large enough, then the solution from (4.19) closely approximates the solution that minimizes the underperformance probability.

Markowitz model

We use the Markowitz (1959) model, which yields returns with minimum variance, subject to the expected return being greater than a target. The problem can be formulated as

$$
\begin{aligned}
\min \quad & \mathrm{Var}\left(\sum_{k=0}^{K} \boldsymbol{x}'\boldsymbol{v}^k \tilde{z}_k\right) \\
\text{s.t.} \quad & \mathbb{E}\left(\sum_{k=0}^{K} \boldsymbol{x}'\boldsymbol{v}^k \tilde{z}_k\right) \ \geq \ \tau \\
& \boldsymbol{x} \ \in \ \mathcal{X},
\end{aligned}
\tag{4.20}
$$

where $\mathrm{Var}(\cdot)$ is the variance of a random variable.

Let $\boldsymbol{V}$ denote a $|\mathcal{E}| \times K$ matrix of correlation factors, where the kth column is $\boldsymbol{v}^k$. Let $\boldsymbol{\Sigma}$ denote the covariance matrix of the random factors $\tilde{z}_0,\dots,\tilde{z}_K$. By assumption $\tilde{z}_k$ is independent of $\tilde{z}_i, i \neq k$, hence $\boldsymbol{\Sigma}$ is a diagonal matrix with its kth diagonal element $\boldsymbol{\Sigma}_{k,k} = \mathrm{Var}(\tilde{z}_k)$. Therefore, problem (4.20) is equivalent to

$$\begin{array}{rrcl} \min & \boldsymbol{x}'\boldsymbol{V}\boldsymbol{\Sigma}\boldsymbol{V}'\boldsymbol{x} & & \\ \text{s.t.} & \sum\limits_{k=0}^{K} \boldsymbol{x}'\boldsymbol{v}^k \mathbb{E}\left(\tilde{z}_k\right) & \geq & \tau \\ & \boldsymbol{x} & \in & \mathcal{X}. \end{array} \tag{4.21}$$

Problem (4.21) is a quadratic optimization problem with binary decision variables, which we solve using CPLEX.

Maximization of Roy's safety-first ratio

We consider maximization of Roy's safety-first ratio as the project selection criterion, and formulate the problem as

$$\max_{\boldsymbol{x}\in\mathcal{X}} \ \frac{\mathbb{E}\left(\sum\limits_{k=0}^{K} \boldsymbol{x}'\boldsymbol{v}^k \tilde{z}_k\right) - \tau}{\sigma\left(\sum\limits_{k=0}^{K} \boldsymbol{x}'\boldsymbol{v}^k \tilde{z}_k\right)}, \tag{4.22}$$

where $\sigma(\cdot)$ is the standard deviation of a random variable. For Roy's safety-first ratio optimization problem to be well posed, we assume that the optimum Roy's safety-first ratio is positive and finite. In this case, problem (4.22) can be equivalently reformulated as the following quadratic optimization problem,

$$\begin{array}{lrcl} \max & \mathbb{E}\left(\sum_{k=0}^{K} \boldsymbol{y}'\boldsymbol{v}^k \tilde{z}_k\right) - \tau u & & \\ \text{s.t.} & \sigma\left(\sum_{k=0}^{K} \boldsymbol{y}'\boldsymbol{v}^k \tilde{z}_k\right) & = & 1 \\ & \boldsymbol{y} & = & u\boldsymbol{x} \\ & u & \geq & 0 \\ & \boldsymbol{x} & \in & \mathcal{X}, \end{array}$$

which we solve using CPLEX.

4.5.2 Comparison with benchmarks

Using the guidelines of Hall and Posner (2001), (a) we generate a wide range of parameter specifications, (b) the data generated is representative of real world scenarios, and (c) the experimental design varies only the parameters that may affect the analysis. A *project instance* is specified by a set of projects, deterministic constraints, correlations, random factors, and interactions. We randomly generate 200 project instances with $n = 50$, and a budget constraint which implies that no more than 25 projects can be selected.

We generate interactions by randomly assigning an index from $UI(1, \ldots, 100)$ to each project. We specify that interaction effects $\tilde{r}_\beta = \eta_{|\beta|} \sum_{i\in\beta} \tilde{r}_i$ exist among projects with the same index. We let $\eta_1 = 1$, $\eta_2 = 1.1$, $\eta_3 = 1.25$, and $\eta_{|\beta|} = 1.5$ when $|\beta| \geq 4$. We consider 50 independent random factors, which follow two-point distributions with mean μ_i and standard deviation σ_i, given by $\mu_i = 80+(i-1)\times 2.5$ and $\sigma_i = 10+(i-1)\times 7.5$, $i = 1, \ldots, 50$. To consider skewness, the probability of a high return for random factor i is independently generated as $p_i^H \sim U(0.55, 0.95)$. Given μ_i, σ_i, and p_i^H, the random factors are fully characterized. The total return of project i is $\tilde{r}_i = \sum_{k=0}^{K} r_i^k \tilde{z}_k$. We let $r_i^i = 1$, which specifies that project i mainly depends on its local factor i, and generate r_i^k as $U(-0.5, 0.5)$, for $k \neq i$, which implies the correlation. We let φ denote the ratio of the target to the maximum expected return, which completes the specification of the project instance, where $\varphi \in \{0.6, 0.7, 0.8\}$.

For each project instance, we (a) find solutions using each selection approach, (b) randomly generate a sample of size 50,000 for the 50 random factors, and (c) compute the returns for each selection approach and each sample instance.

To provide a fair comparison between our two models and the four benchmark approaches, we first apply the four criteria that the benchmark approaches directly optimize. These criteria are expected return, underperformance probability, standard deviation, and Roy's safety-first ratio, respectively. We also consider four widely used performance criteria that are optimized neither by the benchmark approaches nor by the models we propose. The first is expected loss relative to the target. Second, normalizing by the probability of a loss, we consider conditional expected loss. Both these criteria are widely used in financial risk management (Embrechts et al. 1997). Finally, we consider the value at risk (VaR), i.e. the threshold loss that the project portfolio does not exceed with a specified probability, at both the 95% and 99% levels (Jorion 2006). A negative value at risk represents the minimum profit that is attainable with the specified probability. Table 4.4 shows the mean performance of each project selection approach, for all eight criteria. Because of the slow convergence of problem (4.19), the program for underperformance probability is terminated after 10 minutes for each instance. This model is solved using CPLEX 11.0 on a ThinkPad T400 computer.

φ	**Selection approach**	Expected return	UP	Standard deviation	Roy's SF ratio	EL	CEL	VaR @ 95%	VaR @ 99%
0.6	URI	12018	18.67%	3830	0.8969	450	2351	-5400	-2380
	Heuristic URI	11961	19.01%	3835	0.8818	463	2373	-5326	-2294
	Expected return	14367	17.50%	6275	0.9438	716	3906	-3428	1623
	UP	13271	16.70%	4849	0.9785	521	3003	-4798	-846
	Markowitz model	8647	47.40%	2392	0.0123	943	1990	-4500	-2583
	Roy's SF ratio	13737	15.40%	5040	1.0371	489	3055	-4948	-868
0.7	URI	13126	24.21%	4546	0.6836	738	3002	-5249	-1647
	Heuristic URI	13117	24.36%	4548	0.6787	743	3008	-5236	-1628
	Expected return	14367	23.39%	6275	0.7079	1007	4183	-3428	1623
	UP	13646	23.60%	5239	0.6974	849	3527	-4470	-169
	Markowitz model	10082	47.43%	2968	0.0091	1174	2476	-4923	-2536
	Roy's SF ratio	14061	22.09%	5454	0.7537	817	3593	-4539	-117
0.8	URI	13884	30.84%	5243	0.4643	1166	3743	-4764	-570
	Heuristic URI	13865	31.00%	5259	0.4593	1179	3764	-4712	-497
	Expected return	14367	30.49%	6275	0.4719	1393	4494	-3428	1623
	UP	13682	32.73%	5467	0.4031	1335	4022	-4115	360
	Markowitz model	11521	47.41%	3683	0.0080	1459	3077	-5113	-2133
	Roy's SF ratio	14242	29.89%	5768	0.4895	1248	4113	-4173	492

(Columns Expected return through VaR @ 99% are grouped under the header **Criterion**.)

UP=Underperformance probability; SF=safety-first; EL=Expected Loss; CEL=Conditional Expected Loss; VaR=Value at Risk.

Tab. 4.4: Performance of Various Project Selection Approaches.

Table 4.4 shows that the average performance of the URI and Heuristic URI is almost identical across all criteria. Hence, almost the full benefit of our models can be achieved by a simple spreadsheet based solution approach.

We compare the URI against the four benchmark approaches using the mean performance for the three target levels. Comparing with the risk neutral maximization of expected return approach shows that the URI is risk averse, as evidenced by a 27.6% smaller standard deviation, a 24.4% smaller expected loss, and a 49.9% smaller VaR @ 95%. Comparing with minimization of underperformance probability shows that the URI is 12.3% smaller in standard deviation, 16.0% smaller in expected loss, and 15.5% smaller in VaR @ 95%. Comparing with the highly risk averse Markowitz model shows that the URI has a 48.2% smaller underperformance probability, a 70 times greater Roy's safety-first ratio, and a 34.2% smaller expected loss. Comparing with Roy's safety-first ratio maximization shows that the URI is 16.2% smaller in standard deviation, 15.5% smaller in conditional expected loss, and 8 times smaller in VaR @ 99%. In general, the URI provides solutions with high Roy's SF ratio, low expected loss and conditional expected loss, and low Value at Risk. Therefore, with respect to achieving targets, the URI outperforms the benchmark approaches.

The only comparisons where our models fall substantially short are in expected return against the risk neutral maximization of expected return approach, and in minimization of standard deviation against the highly risk averse Markowitz model. This occurs because the URI and Heuristic URI are mildly risk averse. The URI and Heuristic URI provide much better expected loss and VaR performance than maximization of expected return, and much better expected return and underperformance probability than the Markowitz model.

4.5.3 Sensitivity analysis

It is of interest to potential users of our selection models to know when their relative advantage in performance over the benchmark approaches is greatest. The relative advantage of the URI is sensitive to the target level. Compared to other selection approaches, both the disadvantage and advantage of the URI decrease with the target level. The reason is that as the target level increases, the project selections from the URI, underperformance probability minimization, Markowitz model, and Roy's safety-first ratio maximization approaches become less risk averse. Consequently, their project selections become more similar to those from the risk neutral maximization of expected return approach. Hence, the differences among the solutions from these models decrease.

The relative advantage of our models is also sensitive to the amount of interaction. We group the 200 project instances by the number of project bundles: low interaction (at most 62 bundles), medium interaction (from 63 through 66), and high interaction (at least 67). For each selection approach, Figure 4.1 shows the mean performance for each of the three groups, normalized by that of the URI. For the VaR, we evaluate the performance using the maximum expected return as a reference point. The advantage of the URI compared to Roy's safety-first ratio maximization approach, when measured by standard deviation, expected loss, conditional expected loss and VaR, increases with the density of interaction. Similar results apply to comparisons with the underperformance probability minimization approach. Therefore, our recommendation to use the URI over the other two approaches is stronger in project selection environments with more interactions.

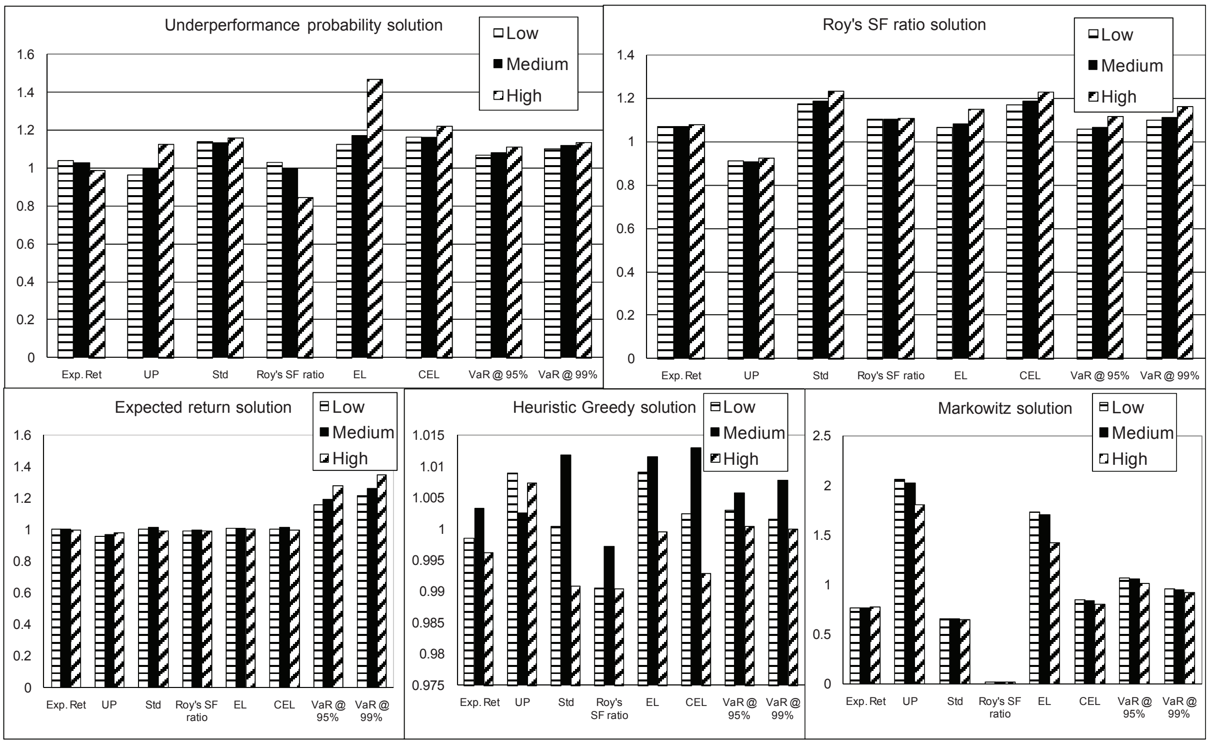

Fig. 4.1: Performance Profiles at Various Interaction Densities.

4.5.4 Robustness

We study the robustness of our URI model, using two computational tests. In the first test, we compare two URI solutions; the first uses full information about the distribution, whereas the second uses only knowledge of the bounded support and mean. We consider the same 200 instances as in Section 4.5.2, except for the uncertain factors. Under full information, the 50 uncertain factors follow the beta distribution with parameters α_i, β_i, $i = 1, \ldots, 50$, which are generated as $U(0.1, 0.9)$; under distributional ambiguity, we calculate the corresponding bound support $[0, 1]$ and mean support $\frac{\alpha_i}{\alpha_i+\beta_i}$ for each uncertain factor. We set $\varphi = 0.7$. For each project instance, we (a) find URI solutions from the two information sets, (b) randomly generate a sample of size 100,000 for the 50 random factors following the beta distribution described above, and (c) compute the returns for both solutions from each information set.

Among the 200 project instances, there are 119 instances where the distributional ambiguity solution is the same as the full information solution. Hence, we show average performance only over the remaining 81 instances, in Table 4.5.

Distributional information	**Criterion**						
	Expected return	UP	Standard deviation	EL	CEL	VaR @95%	VaR @99%
Full	49.07	4.422%	7.024	0.1431	2.854	-37.42	-32.60
Robust	48.88	4.339%	7.022	0.1477	2.887	-37.23	-32.38

UP=Underperformance probability; EL=Expected Loss;
CEL=Conditional Expected Loss; VaR=Value at Risk.

Tab. 4.5: Robustness of URI Performance.

The difference in performance between the two solutions is less than

1.5%, except for expected loss where it is 3.6%. We therefore conclude that the performance of our models is highly robust against distributional ambiguity.

We also conduct a second computational test of the robustness of our model, using real data. The data used is the daily returns of the 49 industry portfolios provided by Fama & French [1]. Our problem is to choose 10 of the 49 industries to achieve a target of 70% of the maximal expected return.

For simplicity, we assume that the returns from these industry portfolios are independent from each other. We use the daily return for dates before January 2011 as historical data, from which we consider two approaches based on different distributional information: 1) Empirical distribution, and 2) Robust approach with support and mean inferred from empirical data. After calculating the optimal portfolio selection for each approach, we then use the daily return data in 2011 to test the performance of the two portfolios. We vary the length of the empirical data used, and summarize the results in Table 4.6.

Length of historical data		**Criterion**						
	Distributional information	Expected return	UP	Standard deviation	EL	CEL	VaR @95%	VaR @99%
2006-2010	Empirical	0.1112	46.03%	17.84	6.632	14.41	29.81	64.20
	Robust	0.2243	48.02%	19.19	7.103	14.79	31.56	65.66
2007-2010	Empirical	0.0317	46.43%	17.26	6.611	14.24	27.82	65.10
	Robust	0.0877	47.62%	17.60	6.665	14.00	28.52	65.29
2008-2010	Empirical	0.1783	46.83%	17.11	6.404	13.68	27.58	64.86
	Robust	0.2962	46.43%	17.88	6.618	14.25	28.15	63.48

UP=Underperformance probability; EL=Expected Loss;
CEL=Conditional Expected Loss; VaR=Value at Risk.

Tab. 4.6: Robustness for Fama & French 49 Industry Portfolios.

Although both approaches have comparable level of risk, the robust ap-

[1] Source: `http://mba.tuck.dartmouth.edu/pages/faculty/ken.french/data_library.html`

proach typically achieves significantly higher expected return, in some cases twice as high, compared to the empirical approach. Using the solutions calculated from the 2008 – 2010 data, Figure 4.2 shows the values of the two portfolios as they evolve over time, assuming that they are both initially valued at \$1. The results indicate that the robust approach outperforms the empirical distribution approach.

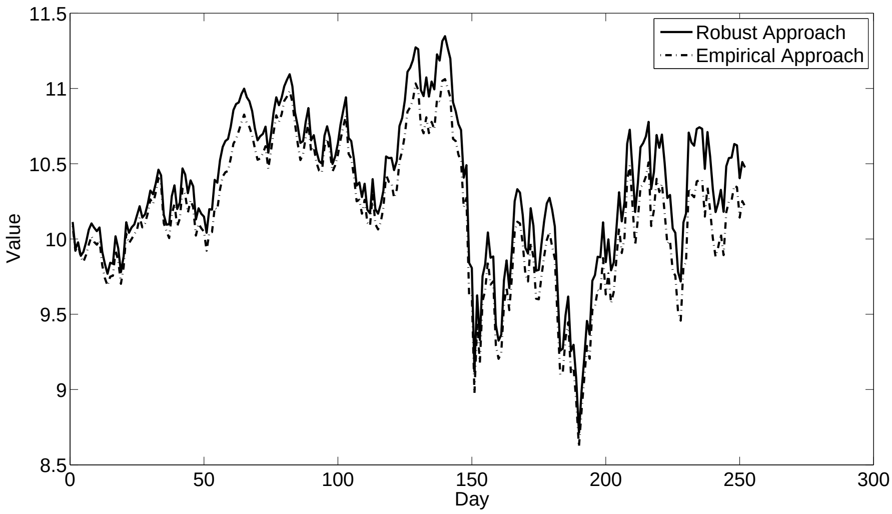

Fig. 4.2: Values of Project Portfolios Evolving Over Time.

4.6 Concluding Remarks

This chapter considers the problem of selecting projects when the return of each project is uncertain. The problem studied is general enough to allow interactions between the different projects, and correlations between their uncertain returns. We describe an underperformance riskiness index for this problem. Our model minimizes the underperformance riskiness index, which is the reciprocal of the ARA parameter, while keeping the certainty equivalent of the uncertain returns above a given target. The model is solved using binary search on the ARA value of the project portfolio, with solution of the subproblems by a Benders decomposition method. We demonstrate computationally that the URI model identifies better project portfolios with respect to achieving the target than those found by classical approaches, including maximization of expected return, mean-variance analysis, minimization of underperformance probability, and Roy's safety-first ratio maximization. For project selection problems that are constrained only by a budget, we describe a simple but highly accurate heuristic URI procedure. The URI procedure also provides robust performance in comparisons with known data and with a sampling approach using real data.

The data requirements of our models are not excessive or unusual. We do not assume knowledge of a specific probability distribution for the factors that affect project return; instead, we assume only bounded support and mean for the factor values. Covariance information is implied by common factors between projects, which should be identified as a risk issue during preliminary project evaluation. Even if covariance information is not fully available, the robust selection model can still be used, based on partial covariance information. Finally, interaction effects between projects are routinely identified during project definition.

Our results provide several insights that managers should find useful. First, it is now possible to design a URI project portfolio that is least risky, subject to meeting a target certainty equivalent level. Second, this design can be achieved very accurately using a computationally efficient procedure.

Third, the resulting project portfolios offer significant benefits over those obtained by all previously used approaches. Fourth, it is possible to balance upside potential and downside risk accurately, by adjusting the target level. Finally, in project selection situations that are constrained only by a budget, a simple spreadsheet-based procedure routinely provides almost exact URI project portfolios.

Several opportunities exist for future research. First, in many practical projects, the initial investment cost is not predictable, and uncertainty about it can be incorporated into a URI model. Second, a related extension is allowing the available budget to be random. In practice, available budgets for funding projects are often uncertain. Third, the URI model should be applied to dynamic project selection problems. In such problems, projects with random investment cost and return become available over time. Consequently, some part of the available budget may need to be held in reserve for future opportunities. Fourth, the problem considered here can be generalized to allow for decisions about the timing of projects, in order to match resource requirements and resource availability over time. A URI approach can usefully be applied to this problem. Finally, it would be valuable to perform large scale behavioral experiments on project selection, to determine the factors that influence how well URI explains those decisions in practice. We hope that our work will encourage future research in these interesting and important directions.

5. CONCLUSIONS

The impact of targets is both observable and sensible in decision making process in industry. It is of great interest to incorporate the targets in the operations management area. In this thesis, we propose a target-based framework for certain operations management problem. The framework does not increase the computational complexity, which is an important issue in practice, especially when problems like dynamic programming and zero-one optimization are involved. Moreover, it is shown to have strong descriptive power and address behavioral preferences observed in laboratory experiments. Last but not least, compared with the abstract concept of risk attitude, which has to be calibrated for adopting expected utility approach, target profit is much easier to observe, and it helps in aligning the whole firm's objective.

In this thesis, we use four risk measures in different contexts (CSM in Definition 1, ESM in Definition 2, CPRI in Definition 4, URI in Definition 10). Among these four measures, CPRI is the only one suitable for multi-period decision problems. With further observation, the CPRI is actually the summation of the URI over all periods. While CSM, ESM, and URI are all measures for single-period risky position, we can see that the maximization of ESM and the minimization of URI are indeed identical. Therefore, ESM and URI are actually equivalent definition, but with different interpretation since they are used in different contexts. CSM and ESM have similar intuition in their definition. The main difference between them is, CSM reflects the emphasis on downside (or upside) risk since it is based on CVaR, while ESM captures the attention on full scale risk as it is based on the certainty equivalent for exponential utility function.

5.1 Future Research

There are several opportunities for future research.

- **General target-based criterion:** As the first step to take into account targets in optimization, in this thesis, we construct the target-based criterion from CVaR and exponential utility function to reduce the complexity in the solution procedure. However, it does not mean that these are the only important target-based criteria. Indeed, it is also of great interest to investigate the impact of target by studying the more general target-based criterion. One potential approach is to construct the target-based criterion from a general utility function and then optimize it.

- **Problems with multiple decision makers:** In this thesis, only one decision maker is involved in all the problems we consider. Nevertheless, in operations management, especially in supply chain management, many important problems are with multiple decision makers, such as the contract design between wholesalers and retailers. If some/all of these decision makers are target oriented when facing uncertainties, it is still unclear how would the coordination be achieved.

- **Verification with real data:** While we are proposing the target-based framework as an alternative to the classical normative model, such as expected utility theory, for operations management, it is desirable to analyze which framework works better than others in what context. Since uncertainties exist, how to come up with a fair comparison between solutions from different framework is a challenging problem. One potential solution may be to use real data from industry to run the back testing.

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